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THE TECHNICAL COLLEGE SERIES

NATIONAL CERTIFICATE MATHEMATICS

VOLUME I

(FIRST YEAR COURSE)

By

P. ABBOTT, B.A. *and* C. E. KERRIDGE, B.Sc.

Completely Revised by

W. E. FISHER, O.B.E., D.Sc. A.M.I.Mech.E.
Formerly Principal, Wolverhampton and Staffordshire College of Technology

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PREFACE

It is now twenty years since the Authors of National Certificate Mathematics set out "to provide a systematic and progessive text-book in Mathematics for students taking mechanical or electrical engineering courses". At that time National Certificates, though still a year or two short of their majority, were well established, and the success of the early schemes had made it clear that preparation for these certificates would be the central activity of the system of part-time technical education favoured in this country—the system of opportunity.

Since the first publication of this work there have been a world war and many years of troubled peace. The total of National Certificate students has increased twenty-fold. Part-time day students now outnumber evening students as greatly as once they were outnumbered. But the three volumes of National Certificate Mathematics still succeed in fulfilling the Authors' original purpose: to meet the immediate practical needs of the first-year no less than of the third-year technical student while providing at all stages a sound basis for more advanced studies.

Why, then, a new edition? Simply because advances in technology have been reflected in changes in courses. National Certificates themselves are responsible for a whole literature of new applications of age-old truths. The day-to-day experience of the typical student has changed. Because of this it has seemed desirable to introduce, here and there in the text, but more especially by way of exercises, matter taken from current National Certificate schemes and examinations.

I wish to renew the acknowledgments made in the first edition to a number of examining bodies which now includes, as well as the City & Guilds of London Institute, the Union of Lancashire and Cheshire Institutes, the Union of Educa-

tional Institutions, the Northern Counties Technical Examinations Council, and the East Midland Educational Union, for permission to print questions which have been set in their examinations. The present edition contains also, by kind permission of many Principals of Colleges, and Chief Officers of Education Authorities, a large number of examples taken from the internal sessional examinations of courses approved for the award of National Certificates. This generous co-operation is highly appreciated and the place of origin is given at the end of each question quoted. Such miscellaneous exercises occasionally anticipate the work of later chapters. This is not altogether accidental: these exercises test the alertness of the student and may provide an experimental introduction to studies which follow.

W. E. Fisher.
Editor, Higher Technical Series

GENERAL EDITOR'S FOREWORD

The Technical College Series today includes many books which are outstanding in their particular fields, and it is the aim of the publisher to maintain and develop the worthy tradition of the series while meeting in full the increasing needs of technical and scientific education.

An outstanding contribution of the technical colleges to education has been the system of National Certificates under which the Ministry of Education and the colleges work in association with the leading professional institutions. This began with National Certificates in Mechanical Engineering under the aegis of the Institution of Mechanical Engineers. The system has now progressed until the schemes cover practically the whole field of higher technology and applied mechanics. In addition to the Institutions of Civil, Mechanical and Electrical Engineers, the Royal Institute of Chemistry, the Institute of Physics, and the Institution of Metallurgists are all associated with National Certificate Schemes. There are also National Certificates in Building and in Commerce, associated with groups of professional institutions. Though the pattern of National Certificates Courses was originally dictated by the needs and limitations of the evening student, the system of endorsements obtainable by further study has now brought about the result that these courses have been extended to meet the full requirements of practice in the subjects with which they deal. During recent years the system of part-time day release of apprentices and learners has become common in all branches of industry as well as in the public services, and now the development of sandwich courses leading to the Diploma in Technology and to the Higher National Diploma is proceeding rapidly.

This has effected something like a revolution in technical education; more time is available for a broader study of the

subjects than was ever possible when almost all technical college students were restricted to evening classes. During the last few years the major professional institutions have taken advantage of these changes and raised their academic standards. A much more fundamental knowledge of the elementary parts of the subjects is now expected. The books included in the Series will be planned to reflect these changes and to provide the part-time and full-time student, working in technical colleges, with text-books designed as an essential aid to the teaching he receives. At the same time these books will form the nucleus of the student's working library which he will require throughout his career.

The increase of day release has also been extended to those apprentices whose academic standard is not sufficiently high to enable them to tackle National Certificate courses, and there are a number of books in the Series directed specifically towards the wide variety of City and Guild examinations available for craft apprentices.

E. G. STERLAND

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CHAPTER 1

REVISION EXERCISES IN ARITHMETIC*

Section A. Percentages

1. Find the values of

(1) 2·5% of 64. (3) 22·8% of 16·2.
(2) 18% of 1160. (4) 8·5% of £7 10*s*.

2. Express the following fractions as percentages (to two places of decimals where necessary):

(1) $\frac{5}{16}$. (3) $\frac{7}{15}$.
(2) $\frac{3}{8}$. (4) $\frac{16\cdot5}{90}$.

3. Find what percentage (to two places of decimals where necessary):

(1) 17 is of 40. (3) 16·5 is of 45·8.
(2) 9·5 is of 128. (4) £2 15*s*. is of £5 10*s*.

4. (1) If 8% of a number is 28·5, what is the number? (2) If 6·5% of a weight is 9·75 gm, what is the weight?

5. From a coil of wire 110 ft long, it is required to cut off $22\frac{1}{2}$%. What length is this, to the nearest foot?

6. If bronze contains 94·5% of copper, what weight of copper will be required for 5 cwt of bronze?

7. Cordite contains 65% of gun-cotton, 30% of nitro-glycerine and 5% of mineral jelly. What weight of each is there in 58 lb of cordite?

* Instead of working these classified exercises, students may proceed direct to the miscellaneous exercises which commence on p. 24. All of these have been taken from examinations held in connection with National Certificate Courses.

8. In a testing machine a wire 84 in. long was extended by 6·58 in. before it broke. What percentage of its original length was this?

9. The efficiency of a certain joint is given by the fraction $\frac{59 \cdot 5}{165}$. What percentage is this?

10. The efficiency of a certain screw is given by $\frac{0 \cdot 035(1 - 0 \cdot 0014)}{0 \cdot 035 + 0 \cdot 038}$. What percentage is this?

11. A number increased by 15% of itself amounts to 713. What is the number?

12. To a certain weight of tobacco, 8% of its own weight of water is added. If the weight is then 40·5 lb, what was the original weight?

13. The workers in a factory are 235 men, 171 women and 29 young persons. State these as percentages of the whole number of workers.

14. A lump of alloy contains 3·41 lb of copper, 0·97 lb of zinc, 0·31 lb of lead and 0·26 lb of other material. What are the percentages of copper, zinc and lead in the alloy?

Section B. Ratio

1. Find, in the simplest fractional form, the ratios of—

(1) 17*s*. 6*d*. to £5.
(2) An inch to a centimetre (1 metre = 39·4 in.).
(3) One mile per hour to 1 ft per sec.
(4) 1 pound to 1 kilogram (1 kg = 2·2 lb).

2. Express the following ratios as decimals:

(1) 4*s*. 6*d*. : 3*s*. 9*d*. (3) 5*s*. 6*d*. per lb : $3\frac{1}{2}$*d*. per oz.
(2) 5 lb : 150 oz. (4) 1·5 : 2·45.

3. A hundredweight of bronze contains 97·28 lb of copper and 4·72 lb of tin. Find the ratio, in decimal form, of the copper to the tin.

4. The mechanical advantage of a machine is the ratio of the resistance to the effort. Find the mechanical advantage when an effort of 96 lb just overcomes a resistance of 1450 lb.

5. Which is the greatest and which the least of the following ratios:

$$5 : 16; \quad 7 : 32; \quad 25 : 81?$$

6. The sides of a triangle are in the ratio of 3 : 4 : 5. The longest side is 6·5 in. Find the other sides.

7. In a certain alloy 68% is copper, 19·8% is zinc and the rest is other metal. Find the ratio of the copper to the zinc.

8. A line 7·2 in. long is decreased in the ratio 2·5 : 1·5. What is its new length?

9. Two pieces of bar iron have cross-sections 2·5 in. square and 5·5 in. by 1·2 in. Find the ratio of their weights per ft run.

Section C. Proportion

1. If the following numbers are in proportion, find x in each case:

(1) $x : 15 = 40 : 7$. (3) $18 \cdot 5 : x = 6 \cdot 8 : 14 \cdot 3$.
(2) $6 : 27 = 30 : x$. (4) $x : 80 = 55 : 64$.

2. If the carriage of 48 tons of coal costs £15, what should be the cost of carrying 72 tons at the same rate?

3. A plank of wood 18 ft long costs 12*s*. $7\frac{1}{2}$*d*. Find the cost of two planks, each 15 ft long, at the same rate.

4. The cost of a casting weighing $4\frac{1}{2}$ cwt is £4 19*s*. What would be the cost of a casting weighing $7\frac{1}{4}$ cwt at the same rate?

5. Find x in the following cases:

(1) $5 : x = x : 80$. (2) $3 : x = x : 147$.

6. Find a mean proportional between 1·5 and 13·5.

7. If a motor consumes $1\frac{7}{8}$ gal of petrol on a run of 24 miles, how much will it consume on a run of 100 miles under similar conditions?

8. For 50 yd of wire netting I had to pay 10*s.* 5*d.* What should I pay for 130 yd at the same rate?

9. A photograph, $7\frac{1}{2}$ in. by $6\frac{1}{2}$ in., is to be enlarged. If in the enlargement the longer side is to be 18 in., what will be the length of the smaller side?

Section D. Approximations

1. It frequently happens that when we are using a number consisting of several figures we do not require to state all the figures: it is sufficient for our purpose to neglect some of the less important, and so use what we call an **"approximate number."** Thus if a man bought a house for £2522, he might speak of it as costing him about £2500. This omission of figures and replacing them by zero is a frequent thing in daily life. The same process is employed in mathematics, but with more precision and under certain rules.

Do not think of an approximate number as embodying a rough value. The word literally means a value very *close* to the correct one.

Frequently a statement is made more correct by omitting superfluous figures. For example, 1 metre is equal to 39·37 in. Multiplication gives the result that 75 metres is equal to 2952·75 in. But the arithmetic which gave us six figures instead of four has added no corresponding accuracy. The six-figure result is actually misleading. The best answer is "2953 in. correct to four figures." See below.

2. Significant Figures

In the example of the house the two figures which we retain—2 and 5—are termed *significant figures.* Let us consider a more difficult example. In 1951 the population of Great Britain was estimated to be 48,871,339. If we wished to state merely the number of millions—*i.e.* use two significant figures—it would be more correct to say the population was 49 millions rather than 48 millions, since the actual population is nearer 49 than 48 millions. Accordingly if we write the number down as approximately 49,000,000, the number is said to be **correct to two significant figures.** Similarly if the number were required **correct to three significant figures**—*i.e.* to the nearest hundred thousand—we should write it down as 48,900,000.

The general rule which we use is as follows:

When a number is to be obtained correct to a required number of significant figures, then if the first of the figures which are discarded is five or greater, the previous figure is increased by one.

In the case of a decimal fraction—*i.e.* a decimal which does not follow an integer—such as 0·05904, the rule is that *the first figure after the decimal point which is not a zero is the first significant figure.*

Thus the number 0·0**5904** contains four significant figures, which are shown in heavier type. It should be noted that if this number were required correct to *three* significant figures, or four decimal places, it must be written as 0·0590. The zero after the 9 must be retained as being a significant figure. To omit it would mean that we should have no information whatever concerning the fourth place of decimals.

3. Accuracy of Answer

If an answer is required correct to a specified number of significant figures, it will always be necessary that the final result of the operations shall contain at least one more figure than is required in order that a correct approximation may be made.

4. Errors Due to Approximation

When the numbers which are employed in arithmetical operations are exact numbers, it is usually easy to obtain an answer to any required degree of accuracy. But when the numbers which we use are themselves approximated, errors must arise from the use of them. Consequently the degree of accuracy which we can reach in the final result is limited, and as a rule must be carefully ascertained.

Thus if we are using the number 3·14, knowing that it is correct to three significant figures, then, by the rules of approximation given above, we know that the correct number is greater than 3·135 and less than 3·145. Consequently the error in the number may be as much as 0·005.

Similar errors occur in cases of measurement. No measurement can be made with absolute accuracy, but we can usually determine the limits between which the accurate results lie. Thus if we measure the length of a line correct to the nearest tenth of an inch and give it as 15·7 in. approximately—*i.e.* three significant figures—then we mean that the true length of the line is greater than 15·65 in., and less than 15·75 in., and consequently the maximum error in the given length is 0·05 in.

When numbers which are approximately correct are employed in calculations, the errors which they involve will clearly affect the final results. Thus if the approximate number 3·14, mentioned above, is multiplied by 9, we get

$$3{\cdot}14 \times 9 = 28{\cdot}26$$

but since the number lies between 3·135 and 3·145, then the product lies between

$$3{\cdot}135 \times 9 = 28{\cdot}215$$

and

$$3{\cdot}145 \times 9 = 28{\cdot}305.$$

It is clear that we cannot tell whether the first decimal place is 2 or 3. Consequently the number of significant figures which we can obtain accurately is only two, or the product will be 28 to two significant figures.

It will be seen that we cannot end a set of operations with a greater number of significant figures than are contained in approximated numbers which we have employed. The rule may be stated thus:

If we operate with several approximated numbers, the number of significant figures which can be depended upon in the final result will in general be less than the least number of significant figures given among the numbers employed.

5. Percentage Error

In estimating the effect of an error in measurements or approximate calculations, we must have regard not only to the actual error itself, but also to its relation to the true value.

Thus if the length of a line be given as 2·6 in., correct to the nearest tenth of an inch, then the maximum error is 0·05 in. or $\frac{1}{20}$ in. The ratio of this to the estimated length is 0·05 : 2·6 or 1 : 52. This expressed as a percentage is about 2%. We say that the **percentage error** is 2%.

Again, if the length of a road is given as 252 miles, correct to the nearest mile, then the maximum error is $\frac{1}{2}$ mile. The ratio of this to the estimated length is $\frac{1}{2}$: 252 or 1 : 504. This is roughly 0·2%.

The terms employed may be defined as follows:

The **absolute error** is the difference between the true value and the approximate value. The true value is often not obtainable, but we can usually determine maximum and minimum values between which it lies.

The **relative error** is the ratio of the absolute error to the true value.

The **percentage error** is the relative error expressed as a percentage.

It frequently happens—as, for example, in most measurements—that the true value is not known. In such cases the error may be expressed as a percentage of the estimated value. Similarly for the relative error. For most practical purposes this is sufficiently accurate.

Example. *If the value of* g *is* 32·191 *and it be taken approximately as* 32, *what is the percentage error in so doing?*

$$\begin{aligned}\text{Absolute error} &= 0{\cdot}191\\ \text{Relative error} &= \frac{0{\cdot}191}{32{\cdot}191}\\ \text{Percentage error} &= \frac{0{\cdot}191 \times 100}{32{\cdot}191}\\ &= 0{\cdot}59\%\end{aligned}$$

Exercises in Approximations

1. Write down the following numbers:

   (1) 18·71604 correct to (*a*) four, (*b*) six significant figures.

   (2) 0·0072038 correct (*a*) to six decimal places, (*b*) to three significant figures.

2. The total imports of the United Kingdom for 1955 were £3866,121,665. Express this correct to (1) the nearest £10,000,000; (2) the nearest £100,000; (3) the nearest £1000.

3. Express 39·9984 correct to (*a*) five, (*b*) four, (*c*) two significant figures.

4. In the following numbers the last figure is approximately correct. State the maximum error in each case:

   (*a*) 3·142. (*d*) 39·37 in.
   (*b*) 2·2 lb. (*e*) 189·4 miles.
   (*c*) 5·126 ft.

5. Give suitable approximations to the results of the following:

   (1) $159{\cdot}4 \times 0{\cdot}0037$. (4) $(12{\cdot}05)^2 \times 0{\cdot}052$.
   (2) $18{\cdot}632 \times 0{\cdot}0469$. (5) $\dfrac{4{\cdot}192 \times 8{\cdot}713}{59{\cdot}8}$.
   (3) $(0{\cdot}598)^2 \div 0{\cdot}082$. (6) $0{\cdot}00512 \div 0{\cdot}826$.

6. The sides of a rectangle were measured as 11·3 in. and 8·4 in. to the nearest tenth of an inch. What is the greatest possible error in the perimeter? Between what limits will the perimeter lie?

7. The following occurred in some computations:

$$(1{\cdot}7168 \times 3) + (0{\cdot}9395 \times 6) - (1{\cdot}9138 \times \tfrac{1}{2})$$

If the decimals are correct to four significant figures, what is the greatest possible error in the result?

8. A rectangular piece of metal is measured by means of a steel rule graduated in tenths and fiftieths of an inch; and its size is written down as—

$$6{\cdot}1 \text{ in.} \times 2{\cdot}36 \text{ in.} \times 0{\cdot}48 \text{ in.}$$

Calculate its volume and justify the number of significant figures you include in your answer.

9. π is 3·14159. To how many significant figures is the approximate value $\frac{22}{7}$ correct?

10. If $\sqrt{2} = 1{\cdot}414$ and $\sqrt{3} = 1{\cdot}732$, each correct to four significant figures, find the following correct to as many significant figures as the data allow:

   (1) $\sqrt{2} \times 12$. (3) $\sqrt{3} \times \sqrt{2}$.
   (2) $\sqrt{3} \times 21$. (4) $5\sqrt{3} + 2\sqrt{2}$.

11. A length whose correct value was 5·37 in. was expressed as 5·4 in. What was the percentage error?

Section E. Square Root

Find the values of the following:

1. (1) $\sqrt{11^2 \times 2^2 \times 5^2}$. (2) $\sqrt{121 \times 64 \times 25}$.

2. (1) $\sqrt{(0{\cdot}02)^2 \times 900}$. (3) $\sqrt{2{\cdot}25 \times 1{\cdot}21}$.
(2) $\sqrt{0{\cdot}04 \times 0{\cdot}64}$. (4) $\sqrt{1{\cdot}44 \times 0{\cdot}09}$.

3. Find the square roots of—

(1) 327184. (2) 18225.

4. Find the square roots of the following to two places of decimals:

(1) 3237. (3) 694·372.
(2) 715. (4) 5·19.

5. Find the square roots, to four places of decimals, of—

(1) 0·913. (4) 0·4.
(2) 0·51647. (5) 0·09164.
(3) 0·056137.

6. Find $\sqrt{2}$ to four places of decimals and use it to find the value of $\frac{6\sqrt{2}}{7}$ to three places of decimals.

7. Find the values, to three places of decimals, of—

(1) $\sqrt{3{\cdot}3^2 + 5{\cdot}6^2}$. (2) $\sqrt{10{\cdot}8^2 + 14{\cdot}4^2}$.

8. Find the values, to three places of decimals, of—

(1) $\sqrt{15{\cdot}6^2 - 10{\cdot}8^2}$. (2) $\sqrt{9{\cdot}3^2 - 4{\cdot}7^2}$.

9. If $f = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q^2}$, find f when $p = 6{\cdot}6$ and $q = 5{\cdot}2$.

10. In the formula $v^2 = u^2 + 25t$ find v when $u = 16{\cdot}5$ and $t = 10$.

11. Find the square roots of (1) $\frac{49}{121}$, (2) $\frac{361}{625}$, (3) $22\frac{42}{289}$.

12. Evaluate $\sqrt{4{\cdot}45^2 - 2{\cdot}55^2}$. (U.L.C.I.)

RATIONALISATION

A number such as $\sqrt{2}$, which cannot be expressed by an exact decimal, no matter to how many places we may work it out, is called an *irrational number*. If it is required to divide by such a number the working would be tedious. This division may often be avoided by the method shown in the following example.

Example. *Find the value of* $\frac{3}{\sqrt{5}}$.

If the numerator and denominator of $\frac{3}{\sqrt{5}}$ be multiplied by $\sqrt{5}$, the value of the fraction is unaltered. The denominator will be converted into a rational quantity.

Thus
$$\frac{3}{\sqrt{5}} = \frac{3 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{5}}{5}$$
$$= \frac{3 \times 2{\cdot}236}{5} = \frac{6{\cdot}708}{5}$$
$$= 1{\cdot}342 \text{ to three places.}$$

This process is called *rationalising the denominator*.

Find the values, to three places of decimals, of—

13. (1) $\frac{1}{\sqrt{2}}$. (2) $\frac{2}{\sqrt{3}}$. (3) $\frac{5}{2\sqrt{3}}$.

14. (1) $\frac{\sqrt{3}}{\sqrt{2}}$. (2) $\frac{\sqrt{2}}{\sqrt{5}}$. (3) $\frac{1}{\sqrt{10}}$.

15. $\sqrt{2} + \frac{1}{\sqrt{2}}$.

16. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$.

17. $\frac{1}{\sqrt{18}}$.

18. If the diagonal of a square is equal to the length of the side multiplied by $\sqrt{2}$, find the side of the square when the diagonal is 8 in.

Miscellaneous Exercises

1. The weights of equal volumes of sea-water and fresh water are in the ratio 65 : 64.

(*a*) By how many per cent is sea-water heavier than fresh water?

(*b*) What volume in gallons will 1 ton of sea-water occupy?

(1 cu ft of fresh water weighs 62·5 lb and occupies $6\frac{1}{4}$ gal.) (Handsworth.)

2. A specimen of alloy contains 3·41 lb copper, 0·97 lb zinc, 0·31 lb lead and 0·25 lb other material. What are the percentages of copper, zinc, and lead, in the alloy? (Handsworth.)

3. (i) Find the average size of a farm in England and Wales, given—

Average size in acres	30	70	200	500
Number of hundreds of farms	1610	614	660	116

(ii) The pressure on a containing wall is 1·275 tons per sq ft; express this in lb per sq in. to three figures. (Sunderland.)

4. Distinguish between "three significant figures" and "three decimal places."

5. A cyclist rides 35 miles at 10 m.p.h. and a further 30 miles at 12 m.p.h. Find the total time his journey takes, and his average speed for the journey. (Sunderland.)

6. On a diagram drawn to a scale of $\frac{1}{4}$ in. $= 1$ ft a rectangle has an area of 12 sq in. What true area does this represent? (W.R. Yorks.)

7. A solution contains $17\frac{1}{2}\%$ of light oil by volume. What volume of solution contains 3 cu ft of light oil? (W.R. Yorks.)

8. (*a*) Convert 43·5 litres into cu in. given that 1 in. $=$ 2·54 cm.

(*b*) Two men invest £5500 and £11,500 respectively in a business, and at the end of the first year a profit of 11·5% is shown. Find what each man will receive in dividends after £1000 has been put aside to cover further capital outlay. (E.M.E.U.)

9. Evaluate correct to three significant figures:

(*a*) $r^2h - R^2h$, where $R = 3{\cdot}152$, $r = 7{\cdot}595$, $h = 14{\cdot}7$.

(*b*) $\dfrac{(471{\cdot}4)^2 \times 21{\cdot}7}{(143{\cdot}9)^3 \times 1{\cdot}053}$.

(*c*) $\dfrac{1}{9{\cdot}1} + \dfrac{1}{0{\cdot}91} + \dfrac{2}{15{\cdot}6}$. (E.M.E.U.)

10. Find the true length of a line if an error of $+1{\cdot}05\%$ was made in measuring it as 179·2 in. (E.M.E.U.)

11. (*a*) Convert 5·3 km into miles given that 39·37 in. are equivalent to 1 metre.

(*b*) The scale of a map is 1 : 100,000 and on it two towns are 2·24 in. apart. What is the actual mileage (to the nearest furlong) between the towns?

(*c*) 1 lb $=$ 453·6 gm. What is the percentage error in assuming 1 lb is only 450 gm? (E.M.E.U.)

12. (*a*) An overhead shaft revolves at 180 r.p.m. A machine belt-driven off this shaft and having a pulley 7 in. diameter is required to be driven at 375 r.p.m. Find the diameter of overhead pulley required.

(*b*) If, in the above case, a driving pulley 18 in. diameter is fitted on the shaft, and a cone of pulleys, 5 in., $6\frac{1}{2}$ in. and 8 in. is fitted on the machine, what range of speeds will this give? (Worcester.)

13. A bronze bearing consists of 74% copper, 19% lead and the rest tin. Find the weight of each metal if the bearing weighs 120 lb. (Sunderland.)

14. The lengths of six rods are 115·8, 116·1, 117·5, 115·9, 116·2 and 116·3 cm. If the length of a rod differs from the average of the six rods by more than 0·82 cm, that rod is to be rejected. Will any of the six be rejected? (Sunderland.)

15. A motorist sets out from a place A at 1.00 p.m. to reach a place 40 miles away for a meeting at 2.15 p.m.: he intends to do the journey at a constant speed. At 1.45 p.m., however, he is held up for 10 min, and then increases his steady speed to arrive in time for the meeting. Plot a graph of distance (vertical) against time (horizontal) for the actual journey, and find his average speed for the journey and his actual speed over the last stage of the journey. (Sunderland.)

16. A man whose stride was 3 ft $1\frac{1}{2}$ in. assumed it to be 1 yd. What percentage error did he make when measuring a cricket pitch by striding out the distance (22 paces)? (W.R. Yorks.)

17. Find the diameter of the driving-wheel of a locomotive if the wheel makes 336 revolutions in 1 mile. (Shrewsbury.)

18. (*a*) A motor-car manufacturer exports on the average 540 cars per month during the first five months of the year, and 690 cars per month during the last seven months of the year. Find his average monthly export throughout the year.

(*b*) When buying an article by hire-purchase it is necessary to pay cash for 15% of the purchase price. For the remainder 8% is added for interest, and the resultant amount is payable in 18 equal monthly instalments. Find the initial cash payment and the monthly payments on an article whose purchase price is £160. (S.W. Essex.)

19. (*a*) A motor race was run on a circuit which measured $7\frac{3}{4}$ miles. The winner's average speed for the whole race of 34 laps was 84·7 m.p.h. His average speed over the first 25 laps was 86·9 m.p.h. Find his average speed over the last nine laps.

(*b*) A certain alloy is made of three metals, A, B and C, in the proportion by weight of 3 : 5 : 11. A casting contains 17 lb of metal A. Find its total weight. (S.W. Essex.)

20. (*a*) A car uses $3\frac{3}{4}$ gal of petrol in 75 miles. How far will it go on 8 gal? How many gallons are needed for 250 miles.

(*b*) A train travels at 40 m.p.h. How long will it take to go 7040 yd?

(*c*) The height of a mountain is given as 3500 metres. If a metre equals 39·37 in. find the height of the mountain in feet. (Worcester.)

21. By making 420 articles a week a man working piece-work earns 5% more than if he were paid at daywork rate. (*a*) How many articles must he make to increase this to $7\frac{1}{2}$%? (*b*) What would he be paid per 100 if the articles are to be sold at £3 2*s*. 6*d*. per 100, the total overhead charges being 150%, and the profit 25% on the total manufacturing cost? (Worcester.)

CHAPTER 2

MEASUREMENT OF AREA

1. Units of Area

The units employed in the measurement of area are derived from those used in the measurement of length.

The area unit is a square whose side is a unit of length.

Thus a square inch is a square each of whose sides is an inch in length. In larger measurements we use a square foot, a square yard or a square mile.

In the **metric system** we may similarly have a square centimetre or a square metre or a square kilometre.

2. Area of a Rectangle

Consider the rectangle ABCD (Fig. 1), in which AB represents on a selected scale 4 in. and CB 3 in. These sides are subdivided into equal parts, each 1 in. long, and lines are drawn parallel to the opposite sides of the rectangle, thus forming 12 squares, each of which has an area of 1 sq in.

It will be seen that there are 4 squares in each row, and 3 of these rows. Consequently the total number of square inches in the rectangle is (4×3), or 12. Or the area of the rectangle is 12 sq in.

Without actually drawing another figure, it may easily be seen that if the length had been 6 in. and the other side 5 in., we should then have 5 rows with 6 squares in a row.

Thus the area would be (5×6) sq in. or 30 sq in.

Treating this more generally,

Let DC contain l units of length,

,, DA ,, b units of length.

Then there are l squares in each of the b rows,

$\therefore$,, ,, $(l \times b)$ squares in all,

i.e. $l \times b$ sq in.

$\therefore$ **The Area is formed by multiplying the length by the breadth.**

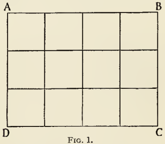


Fig. 1.

Hence if A represents the area, we may write the above result in the form—

$$\mathbf{A} = \boldsymbol{l} \times \boldsymbol{b}$$

If, as is usual, we omit the multiplication sign—

$$\mathbf{A} = \boldsymbol{lb}$$

3. Area of a Square

If in a rectangle the length and breadth are equal, *i.e.* with the notation employed above $l = b$,

then we have

$$A = b \times b$$

or, as we may write it for brevity,

$$A = b^2$$

4. Given the Area, to Find a Side

If the area of a rectangle is known to be 20 sq in. and one side 5 in., then it is clear from the previous working that the other side can be obtained by dividing the area by the given side, or unknown side is $\frac{20}{5} = 4$ in.

Using the result obtained in § 2, we may write—

$$b = \frac{A}{l}$$

and similarly $$l = \frac{A}{b}$$

5. Area of a Triangle

To find the area of the triangle ABC with side BC as a base.

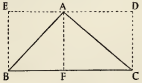


FIG. 2.

Construct the rectangle EBCD so that ED drawn parallel to BC passes through A.

Let AF be the perpendicular from A to the side BC.

Then area of $\triangle$ ABF $= \frac{1}{2}$ area of rect. EBFA,
area of $\triangle$ ACF $= \frac{1}{2}$ area of rect. AFCD,
$\therefore$ Area of $\triangle$ ABC $= \frac{1}{2}$ area of rect. EBCD.

Let BC be b units in length,
,, AF or CD be h units in length.

Then by previous rule—

Area of rect. EBCD $= b \times h$ units of area,
$\therefore$ Area of triangle ABC $= \frac{1}{2}bh$,, ,,

If A represents the area of the triangle, then

$$A = \frac{bh}{2}$$

Example 1.° *A rectangle is divided into two parts as shown in Fig.* 3, *by drawing a line parallel to two sides.*

Let the lengths of the two parts be a and b,
,, ,, ,, other side be x.

To find an expression for the total area of the rectangle.

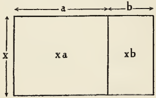


FIG. 3.

First Method.

Considering the areas of the two rectangles separately, these are xa and xb square units respectively.

$\therefore$ If A be the total area of the rectangle

$$A = xa + xb$$

Second Method.

The length of the divided side is $a + b$ units.

$\therefore$ A $= x$ multiplied by $a + b$.

This is not a convenient form. Accordingly we put a

bracket round $a + b$ to show it is to be regarded as one quantity, and so write the formula

$$A = x \times (a + b)$$

or, omitting the multiplication sign, as previously:

$$\mathbf{A = x(a + b)}$$

Since the results obtained by the two methods must be equal, we can write:

$$\boldsymbol{x(a + b) = xa + xb}$$

This is a result of great importance and will be treated more fully in a later chapter.

6. Formula

Such an expression as that which we obtained for the area of a rectangle, viz. $A = lb$, is termed a *formula*. Other examples of formulæ appear on pp. 115–119.

Evaluation of Formulæ

The student will find that formulæ play a very important part in mathematics, as by them it is possible to give a concise, accurate and generalised statement of laws of mathematics or physics.

Example. *Use the formula*

$$L = 2(x + a + b)$$

to find the value of L *when* $x = 10$, $a = 9{\cdot}5$ *and* $b = 4{\cdot}7$.

Substituting these values in the formula, we have—

$$\begin{aligned} L &= 2(10 + 9{\cdot}5 + 4{\cdot}7) \\ &= 2 \times 24{\cdot}2 \\ &= 48{\cdot}4 \end{aligned}$$

If x, a and b are numbers of (say) inches, L will also be a number of inches.

7. Area of a Trapezium

ABCD is a trapezium having AB parallel to DC.

Draw BF perpendicular to DC and DE perpendicular to BA produced.

Let AB = a units of length,
DC = b units of length,
and DE = BF = h units of length.

Then:

Area of $\triangle$ DBC with base $b = \frac{bh}{2}$ or $h \times \frac{b}{2}$ units of area corresponding to the units of length used.

Area of $\triangle$ ABD with base $a = \frac{ah}{2}$ or $h \times \frac{a}{2}$ units of area.

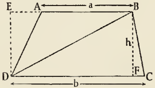


FIG. 4.

Since h is a multiplier of both $\frac{b}{2}$ and $\frac{a}{2}$, we can write the sum of the above expressions in the form

$$h\left(\frac{a}{2} + \frac{b}{2}\right) \quad \text{or} \quad h\left(\frac{a + b}{2}\right)$$

Let the area of the trapezium be A units, then

$$\mathbf{A} = \frac{\boldsymbol{h(a + b)}}{2}$$

Now, $\frac{a+b}{2}$ is the average of a and b. We can therefore state the rule for finding the area of a trapezium as follows:

"Multiply the average length of the parallel sides by the perpendicular distance between them."

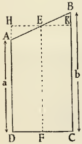


FIG. 5.

Fig. 5 provides another example of a trapezium in which AD and BC are parallel and the angles at D and C are right angles.

The line EF joins the mid-points of AB and DC and is parallel to BC and AD.

Draw HK through E and parallel to DC.

Then EF = HD = KC = $\frac{a+b}{2}$, the average height of the trapezium.

Let DC be h units of length.

Since area of $\triangle$ HAE = area of $\triangle$ BEK, then area of rect. HDCK = area of trapezium ADCB.

$\therefore$ Area of trapezium $= \left(\frac{a+b}{2}\right)h$ as before.

Example. *The area of a trapezium is* 81 *sq in. If one of the parallel sides is* 15·6 *in. and the perpendicular distance between them is* 6 *in., find the other.*

$$\text{Since} \quad A = \left(\frac{a+b}{2}\right)h$$

$$2A = (a+b)h$$

$$\text{that is} \quad 162 = (15{\cdot}6 + b)6$$

$$162 = 93{\cdot}6 + 6b$$

$$68{\cdot}4 = 6b, \text{ or } 6b = 68{\cdot}4$$

$$\therefore \quad b = \frac{68{\cdot}4}{6} = 11{\cdot}4 \text{ in.}$$

8. Area of a Parallelogram

Let ABCD be a parallelogram.

Construct on BC as base a rectangle ECBF, so that CE and BF are perpendicular respectively to AD and DA produced.

Let BC = a units of length, and BF = CE = h units of length.

As in the previous case, the area of the rectangle BCEF = ah square units.

In geometry we can prove $\triangle$ ABF = $\triangle$ ECD

$\therefore$ Area of parallelogram = Area of rectangle
$= ah$

But a = length of one side BC

and h = perpendicular between the opposite sides BC and AD.

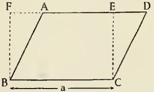


FIG. 6.

Hence—**Area of a parallelogram is equal to the product of one of its sides and a perpendicular drawn to it from the opposite side.**

9. Mid-Ordinate Rule

The method of finding the area of a trapezium by considering the equivalent rectangle can be utilised in determining the area enclosed between a curve and a straight line.

Suppose it is required to find the area of the figure enclosed by the very small part of a curve, AD, the parallel lines BA and CD, and the line BC which is perpendicular to these parallel lines (Fig. 7).

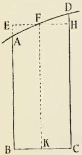


Fig. 7.

Let the perpendicular KF be drawn so that when EH is drawn through F parallel to BC, the area of the portion FDH which is cut off from the curved area is equal to the portion AEF included in the rectangle BEHC. This height KF is the average height of the curve.

Then the area of the rectangle EBCH is equal to the area of the figure ABCD.

In other words, it is the equivalent rectangle. This means that the average height of the curve is the height of the equivalent rectangle.

Suppose it be required to find the area enclosed between the curve AKT and the straight line OS (Fig. 8).

The area is divided up into a number of strips such as OANDE, all of equal width, the total width being OS.

Each strip approximates closely to a trapezium and any mid-ordinate of a strip, such as MN, approximates very closely to the average height of the curve between A and D, if there is no rapid variation in the form of the curve.

With a very large number of strips, the average of all the mid-ordinates will approximate very closely to the average height of the whole curve, and this average height is the height of the equivalent rectangle with OS as its base.

Then the average height of the whole curve thus found, multiplied by the base OS, gives a close approximation of the area between the curve ANKT, the base line OS and the perpendiculars AO and TS.

If SL is the average height, ORLS will be equivalent rectangle for the whole figure.

Since there are as many mid-ordinates as there are strips, we can state the rule for finding the area as follows:

To find the area of the figure, multiply the sum of the mid-ordinates by the width of a strip,

or

Multiply the average mid-ordinate by the total width.

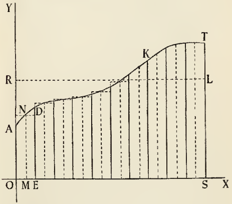


Fig. 8.

10. Application

Sometimes it is necessary, as in engineering, to find the area of a closed curve.

A good example is provided by the Indicator-Diagram or Work-Diagram as shown on p. 38.

The area within the curve (Fig. 9) enables the engineer to

calculate the net work done during one outward and one return stroke of an engine piston.

The length of the figure—*i.e.* PN—represents the length of stroke.

This multiplied by the average mid-ordinate within the curve—that is, of AB, CD, EF, etc.—will give the net work done.

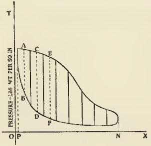


FIG. 9.

In other words, work done may be represented by an area.

11. Symbols normally stand for numbers. In everyday speech and writing, letters are often used just as a short way of writing words. " Let l be the length " and " let A be the area " are examples of the use of letters in this way. But when the letters enter into an algebraic statement or equation they take the place of numbers. In § **7** we said, in connection with the area of a particular rectangle,

$$A = xa + xb.$$

We knew from the information given that x, a and b each stood for a *number* of units of length. If we are told the numbers denoted by x, a and b we can always calculate a

numerical value of A. The number A will be a number of units of area and the actual unit will be the one derived from the unit of length used.

Example. *A rolled steel joist of* **I** *section has approximately the dimensions given in Fig.* 10. *The actual joist differs*

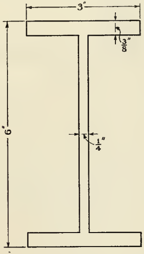


FIG. 10.

slightly mainly because the flanges are tapered to facilitate rolling and the internal angles are rounded. The area of the actual section is given in Engineers' Pocket Books as **3·53** *sq in.* By what percentage have the modifications of the form from Fig. 10 affected the section area?

The two flanges are each 3 in. $\times \frac{3}{8}$ in. They are joined by a "web" which is $5\frac{1}{4}$ in. $\times \frac{1}{4}$ in. The flanges and web have a total cross-section of—

$$2 \times 3 \times \tfrac{3}{8} \text{ in.}^2 + 5\tfrac{1}{4} \times \tfrac{1}{4} \text{ in.}^2$$

$$\begin{aligned} \text{Thus A in.}^2 &= \tfrac{18}{8} \text{ in.}^2 + \tfrac{21}{16} \text{ in.}^2 \\ &= \frac{36 + 21}{16} \text{ in.}^2 \\ &= \tfrac{57}{16} \text{ in.}^2 \\ &= 3\tfrac{9}{16} \text{ in.}^2 \\ &= 3{\cdot}5625 \text{ in.}^2 \end{aligned}$$

or $\quad 3{\cdot}56$ in.^{2} correct to three significant figures.

The tables give the section area as

$$3{\cdot}53 \text{ in.}^2$$

The discrepancy is $0{\cdot}03$ in.^{2}, and comparing this with the calculated value we see that the modifications from the plain rectangular section have led to a reduction of $\frac{0{\cdot}03 \times 100}{3{\cdot}56}\% = 0{\cdot}843\%$ in the cross-section.

NOTE.—(i) The working has been carried to three significant figures because the tabulated section area was stated in this way. It is, however, unlikely that an answer relating to a small difference between two much larger quantities would be *correct* to three significant figures. Can you see why this is so?

(ii) The areas in square inches are obtained by multiplying together two dimensions expressed in inches. Because of this it is common to write in.^{2}, meaning square inches.

12. Volumes Determined by Areas

Most calculations of area can be used directly to give the volumes of simple solids derived from the areas. For example, if a plate of any shape contains A units of area, and its thickness is 1 unit of length, clearly the plate contains A units of volume. If the thickness is t length units the volume is At units. Volumes are dealt with fully in Chapter 12; but this note is introduced in order

that readers may attempt with confidence many miscellaneous exercises taken from examination papers and depending mainly on a calculation of area. Fig. 11 shows such a solid derived from an area, and known as a plate or a prism according as its third dimension is *small* (the thickness of a plate) or *large* (the length of a prism or bar).

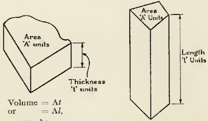


FIG. 11.

Example. *A wall 10 ft × 8 ft has applied to it plaster to a total depth of $\frac{7}{8}$ in. How many cubic feet of plaster are needed?*

The area of wall to be covered is 10×8 sq ft.

$$= 80 \text{ sq ft}$$

The thickness of plaster is $\frac{7}{8}$ in.

$$= \frac{7}{8 \times 12} \text{ ft.}$$

The volume of plaster = Area × Thickness (in corresponding units)

$$\begin{aligned} &= 80 \times \frac{7}{8 \times 12} \text{ cu ft} \\ &= \tfrac{70}{12} \text{ cu ft} \\ &= 5\tfrac{10}{12} \text{ cu ft} \\ &= 5{\cdot}83 \text{ cu ft.} \end{aligned}$$

The answer would best be stated as 5·8 cu ft, to two significant figures, since the thickness of $\frac{7}{8}$ in. could not in practice be exactly attained.

EXERCISE II

I. Find the areas of the following:

(1) A rectangle $11\frac{1}{2}$ in. by $3\frac{3}{4}$ in.
(2) A rectangle 14·3 in. by 17·6 in.
(3) A rectangular field which is 86 yd wide and 103 yd long. Give the answer in acres.
(4) The cross-section of a bar of metal 6·85 in. wide and 4·82 in. thick.
(5) The area of a mile of straight road 27 ft wide.

2. A rectangular garden measures 140 ft by 38 ft. A flower-bed 3 ft wide is made round two sides and one end. What is the area of the remainder?

3. A picture is 3 ft long and $2\frac{1}{2}$ ft wide. The width of the mount between it and the frame is 4 in. Find the area of the mount between the picture and the frame.

4. Assuming no waste, find how many floor-boards 8 ft long and 5 in. wide will be required for a room 24 ft long and $12\frac{1}{2}$ ft wide.

5. What length of carpet from a roll 27 in. wide will be required to cover a room $12\frac{1}{2}$ ft by 15 ft, if there is no waste in the making?

6. What would it cost to paint the walls of a kitchen of length 12 ft, width 11 ft and height 8 ft at 1*s*. 9*d*. per sq yd, on the assumption that area of skirting, doors and fire-place forms $12\frac{1}{2}\%$ of the whole?

7. One side of a rectangular piece of cardboard is 42 cm long, and the weight is 60 gm.

If the weight of a sq cm of the cardboard is 0·24 gm, find the length of the second side.

8. Find the areas of the triangles having the following dimensions:

(*a*) Base 3·2 in., height 11·8 in.
(*b*) Base 124 yd, height 72 yd.
(*c*) Base 8·5 cm, height 11·4 cm.

9. Find the area of an equilateral triangle of side 4 in. by drawing the triangle and measuring its height.

10. A rectangle measures 1 ft 4 in. by 2 ft 3 in. What is the height of a triangle equal to it in area but having a base 3 ft 9 in. long?

11. On squared paper draw accurately a triangle having a base of 4 in. and slant sides $3\frac{1}{2}$ in. and $3\frac{1}{4}$ in. respectively. From the vertex of the triangle draw a line at right angles to the base. Measure the altitude of the triangle. Calculate the area of the triangle and on the same base draw a rectangle of equal area. (U.E.I.)

12. The side of a house 35 ft long and 28 ft high contains four windows each 3 ft by 6 ft. What is the area of the brickwork? (U.L.C.I.)

13 If the press tools are skilfully designed and accurately made a flat-bottomed sheet-metal cup can be produced having the same thickness both of bottom and sides as the circular blank from which it has been drawn.

Assuming this constant thickness, and neglecting the small rounding necessary at the meeting of bottom and sides, *determine the blank diameter* B *in. necessary to produce in thin metal a cup of diameter* D *in. and height* H *in.* Your answer will, of course, be an algebraic formula for B in terms of D and H. Commence your work by making a good-sized clear sketch. (C.G.L.I.)

(In anticipation of Chapters 11 and 12 note that the circumference of a circle is given by the formula $2\pi r$ and the area by the formula πr^2.)

14. Write down the perimeters of the following figures:

(*a*) A square of x in. side.
(*b*) A rectangle of length m in. and breadth n in.
(*c*) A rectangle of length $3a$ in. and breadth $2b$ in.

15. A garden is 150 ft long and 45 ft wide. A lawn in it is a ft long and b ft wide. What is the area of the remainder?

16. How many tiles l in. long and b in. wide would be required to pave a courtyard m ft long and n ft wide?

17. Express p lb weight per sq in. in terms of grams per sq mm, given that 1 in. = 2·54 cm and 1 oz = 28·35 gm. (N.C.T.E.C.)

18. The petrol tank of a motor car is 28 in. long. Its cross-section is a rectangle 10 in. × 6 in. which has all four corners rounded off, the radius of the rounding being 2 in. These are internal dimensions.

(i) Make a good-sized dimensioned sketch of the tank.

(ii) Calculate the number of gallons of petrol that the tank will hold. Assume 277 cu in. to the gallon.

(iii) Calculate the weight of the empty tank if it is made of thin sheet metal (so thin that you need take no account of the thickness) which weighs 0·0056 lb per sq in., all the joints being made without overlap (perhaps by a welding process).

(iv) Give reasons for the number of significant figures you include in each of your answers.

(Based on C.G.L.I.)

19. Find a formula for the area of a regular hexagon inscribed in a circle of radius r, given that the height of each of the six equal triangles into which the hexagon is divided is $\frac{\sqrt{3}}{2}$ times the base of the triangle.

20. A parallelogram has one pair of opposite sides each $7\frac{1}{2}$ in. long, the perpendicular between them being $5\frac{1}{4}$ in. in length. Find the area of the parallelogram.

21. Draw a parallelogram one angle of which is 60°. The sides containing that angle are 3·2 in. and 5 in. in length. Find the height of the parallelogram and calculate its area.

22. An open-topped rectangular container is to be $9\frac{1}{4}$ in. × $7\frac{3}{4}$ in. × $3\frac{3}{8}$ in. high. It could be made from a rectangular piece of sheet metal in two ways: (i) by cutting and joining; (ii) by folding and overlapping (as when wrapping a parcel).

Make good-sized dimensioned sketches in explanation of the alternative methods. The first would be used if the scrap metal cut out were valuable: what would be the percentage saving of metal effected by using method (i) rather than method (ii)?

23. A piece of metal $7\frac{1}{2}$ in. × $5\frac{3}{4}$ in. is to be machined flat by means of a tool which makes repeated cutting and return strokes lengthways of the work. The effective width of the tool is $\frac{3}{32}$ in., and between strokes it is fed crossways of the work by $\frac{1}{32}$ in. How many double strokes will be needed if the tool is just to clear the work on the first and last strokes? Commence by making a good-sized clear sketch.

In setting up the machine an over-run of $\frac{1}{2}$ in. is allowed at each end of the stroke. Express as a percentage the ratio of the area of the metal face machined, to the area "swept" by the tool.

24. A cutting tool is "traversed" along a bar of steel turning in a lathe, and produces a cylinder $2\frac{3}{4}$ in. dia × $11\frac{1}{2}$ in. long. During the finishing cut the traverse of the tool was 1 in. per 75 revolutions, and the actual (tangential) cutting speed was 120 ft per min. How long did the finishing cut take? You may neglect the time needed for the tool to enter the cut.

To keep in step with other operations on the same machine, it is necessary for the above operation to be finished in $4\frac{1}{2}$ min. If the cutting speed may not be

increased, what traverse per revolution is now necessary?

25. The parallel sides of a trapezium are respectively 4·6 in. and 6·7 in., and the distance between them is 3·25 in. Find the area of the trapezium.

26. An angle-iron section has dimensions as shown in Fig. 12.

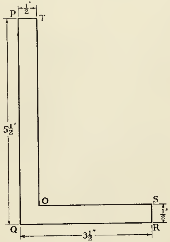


FIG. 12.

(i) Find the area of the section.

(ii) Find the net reduction of area, and express it as a percentage, if the angles at T, O and S are each replaced by a quarter-circle of $\frac{1}{2}$ in. radius.

(iii) Find a formula for the area if PQ is a in., QR is b in., RS is t in.

The miscellaneous exercises which follow have been extracted (by permission) from recent examination papers set in connection with National Certificate Courses.

MISCELLANEOUS EXERCISES

1. The cross-sectional areas of a tree-trunk are as follows:

Distances from end in ft	0	2	4	6	8	10	12
Sectional areas in sq ft	1	3	3·5	4	4·7	5	5·7

Find the volume of timber in the trunk. (Rugby.)

2. (a) Find the volume of metal in a cylindrical pipe of internal and external diameters 6 in. and 7 in. respectively and length 14 ft.

(b) The radius of one circle is twice the radius of another. What percentage is the area of the smaller of the area of the larger circle? (Handsworth.)

3. (a) Express 5° 6′ 18″ in degrees and decimals of a degree.

(b) A rectangular block has a square section of side 5 cm, and is 12 cm long. Find the lengths of the diagonals on all six faces of the block. Find also its volume in cubic metres. (Handsworth.)

4. (a) A rectangle measures 1 ft 4 in. × 2 ft 3 in. What is the height of a triangle of equal area but having a base 3 ft 9 in. long?

(b) What is the external diameter of a pipe 15 sq in. in cross-section if the internal diameter is 8 in.? (Handsworth.)

5. A series of soundings taken across a river channel is given by the following table, x ft being the distance from one shore and y ft being the corresponding depth. Use the mid-ordinate rule to find the area of the section.

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	5	10	13	14	15	16	14	12	8	6	4	3	0

(Nuneaton.)

6. Make up a formula for the volume V of the solid shown in the figure. (Nuneaton.)

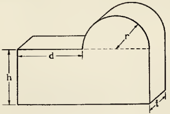


Fig. 13.

7. How many seconds will a train travelling at v ft per sec take to pass completely over a bridge l ft long if the train is x ft long? (W.R. Yorks.)

8. The diagram Fig. 14 shows the end section of a metal casting 8 in. long. Find the weight of the casting if the density of the metal is 0·31 lb per cu in. (Dudley.)

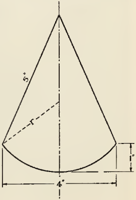


Fig. 14.

9. A chimney is l ft high. Its external diameter is D ft and its internal diameter d ft. Find an expression for V the volume of brickwork in the chimney. Deduce an expression for D in terms of the other symbols.

Find the volume V when D = 8·5 ft, d = 6·5 ft, l = 60 ft. (Sunderland.)

10. Calculate the area of the quadrant of a circle of radius 4 in. by dividing its base into 8 equal parts and setting up 8 mid-ordinates.

Find the area of the quadrant from the formula $A = \frac{\pi R^2}{4}$, and calculate the percentage error in your graphic method. (Coventry.)

11. A regular hexagon is drawn with a side of length a; and a second hexagon is drawn inside the first by joining the mid-points of the sides. A third hexagon is inscribed in the second by the same process. Find the length of side of the third hexagon, and hence the ratio of the areas of the first and third. (Coventry.)

12. A trench is in the form of a trapezium. Its width at the top is 6·5 ft and at the bottom is 3·5 ft, and its depth is 4 ft. If the trench is 40 ft long find the weight of material removed in tons if 1 cu yd of earth weighs 1580 lb. (Handsworth.)

13. (*a*) A path 1 yd wide runs all round a rectangular lawn 20 yd long, 15 yd wide. Is the rectangle formed by the outer edge of the path a similar shape to that of the lawn? Give reasons for your answer.

(*b*) A regular pentagon is inscribed in a circle of radius 5 cm. What is the length of its side? (Handsworth.)

14. 500 lb of material are lifted from a shaft 700 ft deep by means of a rope weighing $1\frac{1}{4}$ lb per ft length. Show by means of a diagram the work done in lifting the material to the surface, and state the total work done. (S.E. Essex.)

15. A garden roller is 2 ft 9 in. in diameter and 3 ft 3 in. wide. What area, to the nearest square yard, does it roll over in 100 revolutions? [$\pi = \frac{22}{7}$.] (Sunderland.)

16. (*a*) A petrol filling-station has three cylindrical storage tanks, each 7 ft diameter and 10 ft long. Find the total storage capacity in gallons if 1 cu ft equals $6\frac{1}{4}$ gal.

(*b*) A swimming-bath, rectangular in plan, is 25 yd long and 15 yd wide. It deepens uniformly, from $3\frac{1}{2}$ ft at the shallow end to $7\frac{1}{2}$ ft at the deep end. Find its capacity in gallons. (Worcester.)

17. A piece of copper 1 ft long, 4 in. wide and $\frac{1}{2}$ in. thick is drawn out into wire of uniform diameter of $\frac{1}{16}$ in. Find (*a*) the length and (*b*) the weight of the wire. (1 cu in. of copper weighs 0·319 lb.) (Worcester.)

18. Water is flowing at 3 m.p.h. in a water-main of 14 in. bore. Find the delivery rate in gallons per minute ($6\frac{1}{4}$ gal = 1 cu ft). (Worcester.)

CHAPTER 3

ALGEBRA

1. Symbolic Representation

In the previous chapter on areas, certain letters, a, b, l, h, were introduced to represent lengths of lines in terms of some unit, and A was taken to represent a number of units of area.

Our purpose here is to show how the operations performed in Arithmetic can be carried out when we generalise and employ letters in place of numbers to represent the magnitude of quantities of any kind. Thus we can speak of x shillings, m pence, l tons or y persons.

Algebra is concerned with such generalised numbers, and the fundamental idea in Algebra is, that we can operate with the symbols just as we do with numbers in Arithmetic. They are subject to the same laws.

2. Multiplication

Just as 5 shillings $= 5 \times 12$ pence $= 60$ pence,

so x shillings $= x \times 12 = x \,.\, 12$, or, as it is usually written, $12x$ pence.

This cannot, as in Arithmetic, be evaluated further, until a definite value is given to x.

The area of 12 rectangles of length h and width b will be $12 \times bh$—that is, $12bh$.

If there are n such rectangles, their total area will be $n \times bh$ or nbh, the units of area being those derived from the length units used.

The area of a square whose side is x in. is $x \times x = x \,.\, x$, or, as we usually write for brevity, x^2 sq in.

The area of n such squares is nx^2 sq in.

The area of n triangles of base b and height h will be $n \cdot \frac{bh}{2}$ or $\frac{1}{2}nbh$.

This method of expressing products can be extended to any number of letters.

Thus $abcd$ means $a \times b \times c \times d$.

a^2bc means $a \times a \times b \times c$.

In the last two examples it will be observed that the symbols a, b, c are used without reference to any quantities such as length, area, etc.

We are thus using the symbols in a completely general sense.

3. Division

Referring again to the rectangle in the last chapter, we had $h = \frac{A}{b}$, which expressed the fact that the area is divided by the base to give the height.

If 24 pence are divided among 8 boys, each boy receives $\frac{24}{8} = 3$ pence.

If 24 pence are divided among x boys, each boy receives $\frac{24}{x}$ pence, and, further, if there be y pence instead of 24, each boy receives $\frac{y}{x}$ pence.

Similarly $\quad a \div b$ is written $\frac{a}{b}$

$ad \div b$ is written $\frac{ad}{b}$

4. Power and Index

The quantity $a \times a \times a$ or aaa or a^3 is called the third **Power** of a, and the small figure 3 which denotes the number of factors in it is called its **Index**.

The index thus indicates the number of times which a occurs as a factor. Similarly $a \times a \times a \times a$ or a^4 is called the fourth power of a and 4 is the index.

5. Products of Powers of the Same Quantity

By our definition $\quad a^2 = a \times a$

also $\quad a^3 = a \times a \times a$

Hence

$$a^2 \times a^3 = (a \times a) \times (a \times a \times a)$$
$$= a^5, \text{ since it is the product of 5 } a\text{'s.}$$
$$\therefore \quad a^2 \times a^3 = a^{2+3}$$
$$= a^5$$

Similarly

$$a^4 \times a^5 = (a \times a \times a \times a) \times (a \times a \times a \times a \times a)$$
$$= a \times a \times a \times \ldots \text{ to 9 factors}$$
$$= a^9$$

It will be seen that similar results will follow, whatever integral indices are employed.

Hence—**When two powers of the same quantity are multiplied together, the index of the product is equal to the sum of the indices of the two powers.**

Extending this idea,

$$(a^2bc)^3 = a^2bc \times a^2bc \times a^2bc$$
$$= (a \times a \times b \times c) \times (a \times a \times b \times c) \times (a \times a \times b \times c)$$
$$= a^6b^3c^3$$

Otherwise

$$(a^2bc)^3 = a^2bc \times a^2bc \times a^2bc$$
$$= a^{2+2+2} \times b^{1+1+1} \times c^{1+1+1}$$
$$= a^6b^3c^3$$

Example 1. $a^3b \times ab^2 = (a \times a \times a \times b) \times (a \times b \times b)$

$$= a^4 \times b^3$$
$$= a^4b^3$$

Example 2. $7xy \times 8x^3y^5 = 7 \times 8 \times x^4 \times y^6$

$$= 56x^4y^6$$

6. Division of Powers of the Same Quantity

(1) Now, $a^3 \div a^2 = \frac{a^3}{a^2}$

$$= \frac{a \times a \times a}{a \times a}$$

$$= a \text{ (on cancelling)}$$

(2) Also $x^5 \div x^2 = \frac{x^5}{x^2}$

$$= \frac{x \times x \times x \times x \times x}{x \times x}$$

$$= x^3$$

No. 1 of the above examples could be written

$$a^3 \div a^2 = a^{3-2} = a^1 \text{ or } a$$

Hence—"**In order to obtain the result of the division of two powers of the same quantity, the index of the divisor must be subtracted from the index of the dividend.**"

NOTE.—The student should note that in the above examples, the index of the dividend is greater than the index of the divisor. The cases in which this is not so will be dealt with later.

7. Coefficient

The algebraic quantity $3abc$ is the product of four quantities, and each of them is a factor of the whole expression.

Any **one** of the four is said to be the coefficient of the product of the other three, when for any purpose we are considering the product of the three as a separate quantity.

Thus 3 is the coefficient of abc,
b is the coefficient of $3ac$.

In $\frac{bh}{2}$, the coefficient of bh is $\frac{1}{2}$.

In $\frac{h}{2}(a + b)$, $\frac{h}{2}$ is the coefficient of $(a + b)$.

Multiplication of certain quantities taken as a group may be indicated by placing the group inside a bracket with the multiplier outside.

Thus the quantity $h(a + b + c)$ indicates that everything in the bracket has to be multiplied by h.

h is the coefficient of $(a + b + c)$ (see Chap. 2, p. 32).

When adding or subtracting multiples of the same quantity we add or subtract the coefficients.

Thus $2ab + 3ab + nab = (2 + 3 + n)ab$
$= (5 + n)ab$

8. Addition of Two or More Quantities

The result of adding one quantity to another is called the **Sum**.

We have already had the sum of a and b, which is written $a + b$ or $b + a$.

The sum of a, x and y is expressed by $a + x + y$.

9. Terms of an Expression

The terms of an expression are the quantities in it which are separated by positive or negative signs. In the expression $3m^2 + 2ny$, $3m^2$ is one term, $2ny$ is another.

This expression has two terms, and so is called a **Binomial Expression.**

The expression $(2a - 3b + 4c)$ consists of three terms, and in consequence is called a **Trinomial Expression.**

10. Reciprocal

The fraction $\frac{1}{7}$ is said to be the reciprocal of 7.

$\frac{1}{x^2}$ is the reciprocal of x^2

ax is the reciprocal of $\frac{1}{ax}$

Thus the reciprocal of any quantity is that quantity divided into unity.

Example. *A horizontal force* F *lb is found to be necessary to cause a weight* W *lb to slide over a flat table. When different weights are used, requiring different forces* F, *it is found that the fraction* $\frac{F}{W}$ *is always the same or nearly the same. The constant value of the fraction* $\frac{F}{W}$ *is generally denoted by* μ *(a Greek letter: pronounced mew).*

Explain why it is usual to describe μ *as the "Coefficient of Friction."*

Since
$$\frac{F}{W} = \mu,$$
$$F = \mu W,$$

so that μ is the coefficient or multiplier which enables us to calculate the friction F from the contact force W.

The Student might now work Ex. III, Sect. A.

11. Subtraction

We know from Arithmetic that subtraction is the inverse operation to addition. Thus to subtract 4 from a number and then to add 4 is to leave the number unaltered. Similarly to add 4 to a number and then to subtract it again leaves the original number unaltered.

Thus $\qquad 10 - 4 + 4 = 10$

or $\qquad 10 + 4 - 4 = 10$

In each case the original operation is undone by an inverse operation.

But in Algebra when symbols are employed a difficulty arises. Thus $a - b$ can be readily understood, as long as a is greater than b, but ceases to have any intelligible meaning arithmetically if a is less than b, though it is clear that $-b$ is still the inverse of $+b$, as in the definition given above.

i.e. $\qquad a - b + b = a$

or $\qquad a + b - b = a$

In Algebra we generalise, and we must now consider what meaning can be given to such a quantity as $a - b$ when b is greater than a.

12. Positive and Negative Quantities

Consider the following example.

In a certain transaction a man gains 12*s*. In a second transaction he loses 8*s*. On the two transactions he has a net gain of 12*s*. − 8*s*. = 4*s*.

He now engages in a third transaction in which he loses 10*s*. If we continue to use a + sign to indicate gains and a − sign for losses, then after the third transaction his position is indicated by + 4*s*. − 10*s*. = − 6*s*. That is, the minus sign shows that he has now had a net loss of 6*s*.

If now in a fourth transaction he loses 3*s*., his total loss will now be shown by − 6*s*. − 3*s*. = − 9*s*.

We would say algebraically that a **profit,** indicated by a + sign, is a **positive quantity,** and **loss,** indicated by a − sign, is a **negative quantity.**

The terms **positive** and **negative** are thus definitely opposite in their meaning and function, and the operations with a **negative** quantity are the reverse of the operations which a **positive** quantity would imply.

13. Positive and Negative Directions

The terms positive and negative are conveniently employed to indicate **opposite directions.**

Thus if, starting from a given point, distances to the **right** are considered as **positive,** then distances to the **left** would be regarded as **negative.**

Again, if distances vertically upwards are regarded as positive, then distances vertically downwards would be considered negative.

Example 1. To some students a billiards handicap of, say, 100 up may indicate some idea of what we mean by *positive* and *negative* quantities.

For simplicity let us suppose that there are four players, A, B, C and D, handicapped as follows:

A is at scratch.
B owes 15.
C owes 5.
D receives 15.

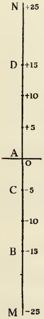


Fig. 15.

We can show the relative positions of the players in the handicap by means of the vertical line MN (Fig. 15).

A neither receives nor owes, and therefore his position is at the zero mark.

D receives a start of 15 from A, and is therefore at plus 15, or $+ 15$.

Since C owes 5 points, he starts with a **debit** of 5—that is, he starts at $- 5$.

Similarly B must start at $- 15$.

It is obvious that the **difference** between the positions of B and D is 30 points.

Again, $- 15$ is clearly below the zero mark, and therefore less than $+ 15$.

If to get the difference we employ the usual method, and subtract the smaller from the larger number, we really have $+ 15 - (- 15)$, which must give 30 points as shown above.

It means that $+ 15 - (- 15) = 15 + 15 = 30$.

Example 2. Diagrammatic Illustration.

If we assume that a movement from left to right is a **positive** movement, then a movement from right to left must be a **negative** one.

If any distance measured along XY (Fig. 16) towards the right and starting from O is taken as a **positive** distance,

a similar distance from O to the left must be considered as a **negative** distance.

Let the length of each division be represented by a, and suppose we wish to find the value of $4a - 3a$.

We move from O to A by 4 steps, and then move back 3 steps to B, and the final position is $1a$ or $+ a$ from O.

To show $2a - 5a$, we move 2 steps from O to the right—that is, to C—and back 5 steps to the left to D, so that the ultimate position is 3 units or steps to the left of the zero or starting point—that is, at $- 3a$.

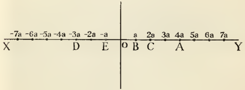


Fig. 16.

Again, starting from C and moving back to B, we perform the operation, for $2a - a$, giving us a.

From B to O shows $a - a = 0$.

From O to E shows $O - a = - a$.

This means that $- a$ is less than zero, or that $- 1$ is one less than zero.

The Addition of Positive and Negative Quantities.

Suppose it is required to find the sum of the following quantities:

$$- 3a, 2a, - 5a, 4a.$$

This means adding them together, but paying due attention to the **sign** of each quantity, which, as far as the diagram is concerned, means direction.

Starting from 0, this would give 3 steps to the left, 2 to

the right, 5 to the left and 4 to the right, finishing ultimately 2 to the left, and thus $-2a$ is the sum of the above quantities.

We could have obtained the same result by taking the two negative quantities first, and then the positive.

To perform the addition arithmetically, the sum of the positive quantities $2a$ and $4a$ is $6a$. The sum of the negative quantities $-3a$ and $-5a$ is $-8a$.

$\therefore$ The total sum is that of $6a$ and $-8a$, or $6a - 8a$, which we write as $-2a$.

To Subtract a Negative Quantity.

As already explained, adding $2a$ and $-3a$ is shown by a movement of 2 steps to the right to A (Fig. 17) and then a

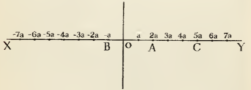


FIG. 17.

movement from A of 3 steps to the left to B, ending with OB, which represents $-a$.

To show $2a - (-3a)$ we must **reverse** the direction for the second movement, which means that we arrive at C, giving us OC, which represents $+5a$.

This is seen to be the same as $2a + 3a$, which equals $5a$. We therefore obtain the following rule:

"To subtract a negative quantity reverse its sign, and proceed as in addition."

14. Brackets

Since in the expression $5a + (a + b - 2a)$, the positive sign in front of the bracket means **no reversal,** it is equivalent to $5a + a + b - 2a$, which is equal to $4a + b$.

If, however, we have $5a - (a + b - 2a)$, the signs in the bracket must be reversed if we remove that bracket.

$$5a - a - b + 2a$$

or

$$6a - b.$$

Example. $(3a - 2b) - (5a + b) - (-8a - 7b)$

$$= 3a - 2b - 5a - b + 8a + 7b$$
$$= 6a + 4b$$

In an example such as $3a - \{2b - (5a + b - c)\}$, we first remove the innermost bracket.

This gives us $3a - \{2b - 5a - b + c\}$.

Then, removing the outer bracket, and proceeding as in the previous case, we get:

$$3a - 2b + 5a + b - c$$
$$= 8a - b - c$$

15. Rules for the Use of Signs in Multiplication and Division

Multiplication and Division with Negative Quantities.

Multiplication.

We know that $5a + 5a + 5a = 15a$.

In other words, **three** times $5a$—that is,

$$5a \times (+3) = 15a$$

Similarly $\quad -5a - 5a - 5a = -15a,$

otherwise $\quad -5a \times 3 = -15a$

It will be observed here that one of the two quantities is negative, while the other is positive, and the result is a negative quantity.

As has already been mentioned, a $-ve$ quantity **operates in the opposite sense** to a $+ve$ quantity.

Hence if $\quad -5a \times (+3) = -15a$

$\quad -5a \times (-3) = +15a$

Again, if $\quad 3a \times 2b = 6ab$

$\quad -3a \times 2b$ must equal $-6ab$

and $\quad -3a \times -2b$ must equal $+6ab$

Example. $4ab \times -2ac \times 5ab = 20a^2b^2 \times (-2ac)$

$$= -40a^3b^2c$$

Division.

1. We have seen above that

$$3a \times 2b = 6ab$$

Then $\quad 6ab \div 2b = 3a$ (the other quantity)

2. Also since $\quad -3a \times 2b = -6ab$

$\quad -6ab \div 2b = -3a$

and $\quad -6ab \div -3a = 2b$

3. Since $\quad -3a \times -2b = 6ab$

$\quad 6ab \div -3b = -3a$

Examining these results we get the following rule:

"In Multiplication and Division, if the two quantities have the same sign, the result is a positive quantity. If the signs are different, the result is a negative quantity."

Briefly we can say

Like signs give +
Unlike signs give −

Examples.

1. $\quad -2ab \times 3bc \times -2ac = -6ab^2c \times -2ac$

$$= 12a^2b^2c^2$$

2. $$\frac{15x^3y^2}{-5xy} = -3x^2y$$

3. Simplify $m(n + p - 2q) - n(m - p - 3q)$

The expression $= mn + mp - 2mq - mn + np + 3nq$

$\quad = mp - 2mq + np + 3nq$

16. The Addition and Subtraction of Simple Fractions with Numerical Denominators

In an example such as

$$\frac{m}{12} + \frac{m+n}{4} - \frac{2n-m}{3}$$

it must be clearly understood that the positive sign in front of the second fraction refers to the fraction as a **whole,** and the same statement applies to the minus sign in front of the third fraction.

As in Arithmetic, the equivalent of each fraction must first be expressed with the same denominator, generally known as a **common denominator,** which in the above case is 12.

Then $\quad \dfrac{m+n}{4}$ becomes $\dfrac{3(m+n)}{12}$

and $\quad \dfrac{2n-m}{3}$ becomes $\dfrac{4(2n-m)}{12}$

Hence
$$\frac{m}{12} + \frac{m+n}{4} - \frac{2n-m}{3}$$
$$= \frac{m}{12} + \frac{3(m+n)}{12} - \frac{4(2n-m)}{12}$$
$$= \frac{m + 3(m+n) - 4(2n-m)}{12}$$
$$= \frac{m + 3m + 3n - 8n + 4m}{12}$$
$$= \frac{8m - 5n}{12}$$

It will be seen that the terms in any numerator must be dealt with as one quantity, and hence the line which separates numerator from denominator really acts as a bracket.

17. The Indicator Diagram Furnishes an Example of the Rule of Signs; also of Positive and Negative AREAS

Engine indicator diagrams (refer Fig. 9, p. 38) generally have marked upon them a line showing atmospheric pressure. A single-acting piston (such as that of most motor-cycle and motor-car engines) has atmospheric pressure applied to one side all the time. Then, within the working space enclosed by the piston, we can regard pressures *above* atmospheric as positive or +, and pressures *below* atmospheric as negative or −. When the piston moves *outwards* (towards the crank) in response to the pressure of the enclosed gas or steam we can regard its displacement as positive; when the piston moves *inwards*, against the pressure of the enclosed working substance, we can regard its displacement as negative.

Fig. 18 (*a*) is copied from an indicator diagram taken from a single-acting engine working upon the four-stroke cycle. It shows the pressure for every position of the piston during the four strokes which are (i) compression, (ii) expansion, (iii) exhaust, (iv) suction. This diagram has to record very high pressures, so on the scale to which it is drawn the lines indicating the exhaust and suction cannot be distinguished from one another or from the atmospheric line. It is customary therefore to obtain another diagram with a much more open pressure scale. This will record the exhaust and suction strokes, but not the higher pressures of compression and expansion. This diagram, often known as a pumping diagram, is drawn in Fig. 18 (*b*).

On Fig. 18 (*a*) and (*b*) four short corresponding lengths of the record for the four engine strokes are shown in heavy

lines and marked respectively 1, 2, 3 and 4. These four lengths are redrawn in Fig. 19. The areas between 1, 2, 3 and 4, and the atmospheric line give interesting examples of the rule of signs in multiplication.

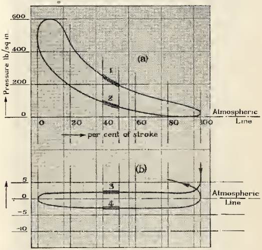


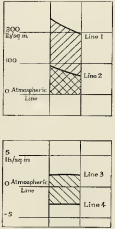
FIG. 18.

Area under 1. Here the pressure is above atmospheric or *positive*, the piston displacement is *positive*. So their product (the area under line 1, representing the work done during a short part of the expansion stroke) is *positive*. Positive × Positive *gives* Positive.

Area under 2. Here the pressure is again above atmospheric, or *positive*, but the piston displacement is inward or

negative. So their product is *negative.* Positive × Negative *gives* Negative.

Area under 3. Here the pressure is still above atmospheric, or *positive*: the piston displacement is again *negative.* So their product is *negative.* Positive × Negative *gives* Negative.



Lines 1 and 2 transferred from Fig. 18 (*b*).
Lines 3 and 4 transferred from Fig. 18 (*a*).

Fig. 19.

Area between 4 *and atmospheric line* 4. Here the pressure is below atmospheric, or negative: the piston displacement is outward or positive. So their product is negative. Negative × Positive *gives* Negative.

The positive area under 1 is shaded /////. The three negative areas are shaded \\\\\. Following the diagram

round, it can be seen that the main area in Fig. 18 (*a*) is traced clockwise. This is a positive area and represents work done *upon* the piston. The loop in Fig. 18 (*b*) is traced anti-clockwise. This is a negative area and represents work done by the piston in drawing in the cylinder charge.

EXERCISE III

Section A

1. Write down answers to the following:

   (*a*) The coefficient of x in $3ax$, $\frac{1}{2}xy$, $x + 5bx$.

   (*b*) The coefficient of ab in $5abc$, $3abx^2$, $\dfrac{5ab}{1+x}$.

   (*c*) The coefficient of x in $xy + xz$.

2. Simplify the following:

$$3x + 5z + 2y + 5x + 6{\cdot}5y + 10z$$

and find the value of this when $x = 1$, $y = 2$, $z = 3$.

3. Multiply together the following:

   (*a*) $3x^2y \times 2xy^2$. (*c*) $2ab \times 3bc \times 4ac$.

   (*b*) $\frac{1}{4}pq \times 3q^2$. (*d*) $\frac{1}{2}at \times \frac{2}{3}bt \times \frac{1}{5}a^2t$.

and find the value of the last when $a = 5$, $b = 6$, $t = 2$.

4. Simplify the following by adding together the fractions:

   (*a*) $\dfrac{a+3b}{2} + \dfrac{2a+5c}{3} + \dfrac{4b+6c}{5}$.

   (*b*) $\dfrac{3}{2a} + \dfrac{4}{3a} + \dfrac{5}{a}$.

5. Evaluate the following:

   (*a*) $28x^4y^2 \div 4x^3y$. (*c*) $\dfrac{2x^2}{y} \div \dfrac{5}{3y^2}$.

   (*b*) $a^3 \div \frac{4}{7}a^2$. (*d*) $\dfrac{6ab}{5cd} \div \dfrac{4a^2}{7bd}$.

6. Write down the reciprocals of the following:

(1) $\frac{2a}{3}$. (2) $\frac{4}{3c^2}$. (3) $\frac{4}{5ax^2}$. (4) $\frac{3}{2}x^2$.

7. Write down the squares of the following:

(a) $14x$. (b) $6a^2$. (c) $\frac{2}{3}x^2y$. (d) $\frac{5x}{3y^2}$.

8. Write down the square roots of the following:

(a) $9x^2y^2$. (c) $\frac{4}{9}x^4y^4$.

(b) $64a^2y^6$. (d) $\frac{16x^2y^2z^2}{9}$.

9. Given that $l_1 = l_0\{1 + \alpha(t_1 - t_0)\}$, find the value of α if $l_1 = 9{\cdot}007$, when $l_0 = 9$, $t_1 = 85$ and $t_0 = 15$; l_1 and l_0 being lengths in inches, t_1 and t_0 temperatures in degrees Centigrade. Explain why the small fraction α is conveniently known as the Coefficient of Expansion.

10. Referring to Fig. 18 (*a*), by means of dividers transfer the lines of the diagram as accurately as possible to your paper, and then, by mid-ordinates or otherwise, find what work is done during each of the four strokes of the cycle, remembering the rule of signs in multiplication. To find the scale of your diagram take the cylinder bore as 4 in. and the piston stroke as 5 in.

From your results find the work done per minute if the engine has four cylinders and is turning at 920 revolutions per minute. You need not take Fig. 18 (*b*) into account.

11. Simplify

$$\frac{\frac{d^5}{d^4} \times \frac{d^3}{d}}{\frac{a}{a^3} \times \frac{a^9}{a^7}} \qquad \text{(U.E.I.)}$$

12. Express in its simplest form each of the following:

(i) $\frac{c^3 \times c^{10}}{c^6 \times c^5}$. (ii) $\frac{(ab)^3}{a^2b^2}$. (N.C.T.E.C.)

SECTION B

13. Write down the answers to the following:

(1) $(+a) - (-a)$.	(9) $(-a) \div (-a)$.
(2) $(-a) - (-a)$.	(10) $(-25) \div (+5)$.
(3) $(-a) - (+a)$.	(11) $(-25) \div (-5)$.
(4) $(+a) \times (-a)$.	(12) $(+25) \div (-5)$.
(5) $(-a) \times (+a)$.	(13) $(-4) \times (+3) \times (-2)$.
(6) $(-a) \times (-a)$.	(14) $(-a) \times (+3a) \div (-2a)$.
(7) $(+a) \div (-a)$.	(15) $\{(-6x^2) \div (-2x)\} \times (-x)$.
(8) $(-a) \div (+a)$.	(16) $(-5x) \times (-2x) \times (-x)$.

14. Add up and simplify the following:

(a) $\frac{3-n}{2} - \frac{n-2n^2}{3} + \frac{3+2n^2}{5}$.

(b) $3x - \frac{2x+3y}{7}$.

(c) $\frac{4a+7b}{6} - (a-4b)$.

(d) $3x - 4z - \frac{2x+5y-6z}{3}$.

15. Find the value of $2(3a - 4b) - \frac{1}{2}(4a - 3b)$, when

$$a = -2{\cdot}5, \quad b = -3{\cdot}5.$$

16. Find the value of $1 - 3x + 5x^2 - x^3$, when $x = -1{\cdot}5$.

17. Supply the quantities missing in the brackets in the following:

(a) $1 - x - x^2 = 1 - x(\qquad)$.

(b) $5x^2 - 7x + 14 = 5x^2 - 7(\qquad)$.

(c) $3a - 2b + 4c + 7a = 10a - 4(\qquad)$.

18. Simplify:

(i) $\frac{3x^2 - 13x + 14}{2x^2 + x - 10} \times \frac{2x^2 + 5x}{3xy - 7y}$.

(ii) $\frac{x+1}{x-1} - \frac{x-1}{x+1} + \frac{4}{x^2-1}$. (U.L.C.I.)

19. Simplify:

(i) $(x^3)^5 \div (x^3)^4 \times x^{-1}$.
(ii) $4[7x - 12 - 3(2x - 3)]$.
(iii) $(2a + 3b)^2$.
(iv) $\dfrac{3}{1 - a} + \dfrac{4}{(1 - a)^2}$.
(v) $(4y + 3)(2y - 4)$. (Shrewsbury.)

20. (*a*) If a train travels at V m.p.h., how long will it take to travel D yd?

A motor travels 100 miles at an average speed of 40 m.p.h., and does the return journey at an average speed of 50 m.p.h. What is the total travelling time?

(*b*) If $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$, find R when $R_1 = 9{\cdot}5$, $R_2 = 6{\cdot}4$ and $R_3 = 3$. (Worcester.)

CHAPTER 4

I. ALGEBRAIC OPERATIONS

1. The Distributive Law

On p. 32 it was shown that

$$x(a + b) = xa + xb$$

That is, the product of x and $a + b$ is found by multiplying each of the terms of $a + b$ by x and adding the result. In other words, the factor x is distributed to each of the terms a and b. This is the fundamental law of Algebra, known as the **Law of Distribution.**

The method employed may be extended to any number of terms,

$$e.g. \quad x(a + b + c) = xa + xb + xc$$

The law can similarly be shown to be true when some of the terms are negative.

Thus $\quad x(a - b - c + d) = xa - xb - xc + xd$

It should be noted that any quantity which is a factor of each term of an expression is also a factor of the whole expression.

Examples.

1. $5n(2a + 3b - 4c) = 10an + 15nb - 20nc$.
2. $3x^2(2x^2 - 5x + 7y) = 6x^4 - 15x^3 + 21x^2y$.

2. Product of Two Binomial Expressions

(1) The product of $(m + n)$ and $(p + q)$

Let ABCD be a rectangle with AD divided at M so that AM $= m$, and MD $= n$ (Fig. 20).

AB is divided at Q so that AQ $= p$, and QB $= q$.

Draw QP parallel to AD.
Draw MN parallel to AB.

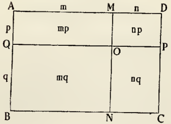


FIG. 20.

Then AD $= m + n$, and AB $= p + q$.

Now, rect. ABCD = rect. AQPD + rect. QBCP
that is, $(m + n)(p + q) = p(m + n) + q(m + n)$
$= pm + pn + qm + qn$

It will be noted that in the expression $(m + n)(p + q)$ each term in one bracket multiplies each term in the other.

Similarly it can be shown that

$$(m - n)(p + q) = mp + mq - np - nq$$
$$(m - n)(p - q) = mp - mq - np + nq$$

(2) The square of a binomial.

Applying the above, it will be seen as a special case that

$$(a + b)^2 = a^2 + 2ab + b^2$$

In words, " the square of a Binomial Expression is the sum of the squares of the two terms taken separately, together with twice the product of the two terms."

Similarly we can show that

$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 - \boldsymbol{2ab} + \boldsymbol{b}^2$$

Example 1. $(ab + xy)^2 = a^2b^2 + abxy + abxy + x^2y^2$
$= a^2b^2 + 2abxy + x^2y^2$

Example 2. $(mn - p)^2 = m^2n^2 - mnp - mnp + p^2$
$= m^2n^2 - 2mnp + p^2$

3. Product of Two Binomials with One Quantity the Same in Each

Applying the above to such a product as $(x + 3)(x + 5)$ we see that

$$(x + 3)(x + 5) = x^2 + 3x + 5x + 15$$
$$= x^2 + (3 + 5)x + 15$$
$$= x^2 + 8x + 15$$

It will be noted in this result that the coefficient of x in the middle term is the sum of the numbers 3 and 5.

We can use this fact and extend it in other examples.

Thus $(x - 9)(x + 7) = x^2 + (-9 + 7)x - 63$
$= x^2 - 2x - 63$
$(x + 8p)(x - 2p) = x^2 + (8p - 2p)x - 16p^2$
$= x^2 + 6px - 16p^2$

In the expression $(3x - 5)(2x + 7)$ the first term in each **binomial** is already a multiple of x.

Then $(3x - 5)(2x + 7) = 6x^2 + (3 \times 7)x - (5 \times 2)x - 35$
$= 6x^2 + (21 - 10)x - 35$
$= 6x^2 + 11x - 35$

4. The Product of the Sum and Difference of Two Algebraic Quantities

Let the quantities be x and a.

We have to find the value of $(x + a)(x - a)$

$$(x + a)(x - a) = x^2 + (+a - a)x - a^2$$

that is, $(\boldsymbol{x} + \boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) = \boldsymbol{x}^2 - \boldsymbol{a}^2$

This can be stated as follows:

"The product of the *sum* and *difference* of two quantities is equal to the difference of their squares."

Example. $(7a + 5b)(7a - 5b) = (7a)^2 - (5b)^2$
$$= 49a^2 - 25b^2$$

Example. $(3mn + 5pq)(3mn - 5pq) = (3mn)^2 - (5pq)^2$
$$= 9m^2n^2 - 25p^2q^2$$

Taking the example $(x + a)(x - a)$, let $x = p + q$.

$$\begin{aligned} \text{Then } (x + a)(x - a) &= [(p + q) + a][(p + q) - a] \\ &= (p + q)^2 - a^2 \\ &= p^2 + 2pq + q^2 - a^2 \\ \text{Now let } \quad a &= m - n \\ \text{Then } (x + a)(x - a) &= [x + (m - n)][(x - (m - n)] \\ &= x^2 - (m - n)^2 \\ &= x^2 - (m^2 - 2mn + n^2) \\ &= x^2 - m^2 + 2mn - n^2 \end{aligned}$$

II. FACTORS

5. An analysis of the illustrations of various products in this chapter, and the methods of obtaining them, afford a means of finding the factors of certain expressions.

A. Expressions which have One Factor Consisting of One Term Only

Example 1. We know that

$$\begin{aligned} 9x - 7x + 11x - 2x &= 20x - 9x \\ &= 11x \end{aligned}$$

Here x is a factor of each term and is also a factor of the whole expression.

We can write the above as follows:

$$\begin{aligned} 9x - 7x + 11x - 2x &= x(9 - 7 + 11 - 2) \\ &= x \text{ (sum of coefficients of } x) \\ &= 11x \end{aligned}$$

Example 2. If we have to factorise $xn - xp - xq$, we notice that there are three terms each containing x, and the coefficients of x are respectively n, $-p$, and $-q$.

The sum of these coefficients is $(n - p - q)$. Hence as in the example above

$$xn - xp - xq = x(n - p - q)$$

This is the converse of the example given on the Distributive Law.

Example 3. In the expression $12ab - 16bc + 2nb$ we have 2 and b as common factors,

$$\therefore \quad 12ab - 16bc + 2nb = 2b(6a - 8c + n)$$

Example 4. In the expression $7p^3q - 14p^4q^3 + 21p^2q^5$, 7, p^2 and q are common factors—that is, $7p^2q$ is a common factor of the whole expression.

$$\therefore \quad 7p^3q - 14p^4q^3 + 21p^2q^5 = 7p^2q(p - 2p^2q^2 + 3q^4)$$

B. Factorising Expressions of Four Terms which can be Expressed as the Product of Two Binomials

6. Consider the expression $pm + np + qm + qn$. In the first two terms p is a common factor.

Then $\qquad pm + np = p(m + n)$.

In the last two q is a common factor.

Then $\qquad qm + qn = q(m + n)$.

We now have

$$pm + np + qm + qn = p(m + n) + q(m + n)$$

In $p(m + n) + q(m + n)$, the sum of the coefficients of $(m + n)$ is $p + q$.

$$\therefore \quad p(m + n) + q(m + n) = (p + q)(m + n)$$

This result is the converse step of the example illustrating the product of two binomials.

Example. *Factorise* $6x^2 - 12xa + 3nx - 6na$.

$$\begin{aligned} \text{Now} \qquad 6x^2 - 12xa &= 6x(x - 2a) \\ \text{and} \qquad 3nx - 6na &= 3n(x - 2a) \\ \text{Then } 6x^2 - 12xa + 3nx - 6na &= 6x(x - 2a) + 3n(x - 2a) \\ &= (6x + 3n)(x - 2a) \end{aligned}$$

C. Factors of Expressions of the Type $ax^2 + bx + c$ in which a, b and c have Numerical Values

7. Case I. When a is unity.

Example 1. *Factorise* $x^2 + 9x + 20$

$$\begin{aligned} \text{This can be written} \quad &= x^2 + 5x + 4x + 20 \\ &= x^2 + 4x + 5(x + 4) \\ &= x(x + 4) + 5(x + 4) \\ &= (x + 5)(x + 4) \end{aligned}$$

It will be noted that the coefficient of x in the original expression is the sum of one pair of factors of 20. These can be determined by inspection.

Example 2. *Factorise* $x^2 - 4x - 45$

$$\begin{aligned} \text{Now} \quad x^2 - 4x - 45 &= x^2 - 9x + 5x - 45 \\ &= x(x - 9) + 5(x - 9) \\ &= (x + 5)(x - 9) \end{aligned}$$

It is evident that the real point is to choose factors of $- 45$ such that their sum gives the coefficient of x in the original expression.

After a time the student can select the binomial factors at sight, and, having obtained them, he can check the result by their multiplication.

8. Case II. When the coefficient of x^2 is not unity.

The method of trial is the best to employ.

When dealing with the product of two binomials (p. 73) we saw that

$$\begin{aligned} (3x + 5)(2x + 3) &= 3x(2x + 3) + 5(2x + 3) \\ &= 6x^2 + (9x + 10x) + 15 \\ &= 6x^2 + 19x + 15 \end{aligned}$$

Our problem now is the converse of this. From the possible factors of $6x^2$ and 15 we have to select two pairs such that the coefficient of x in the product is 19. We see that the x term is obtained in the above from the sum of the products of $3x$ and 3, and $2x$ and 5. The problem is to hit upon this combination. In our trial we must proceed systematically, adopting some such method as the following.

Example 1. *Find the factors of* $15x^2 + 28x + 12$.

Arrange one set of factors $(5x + 3)$ and $(3x + 4)$ as shown in the diagram.

$$5x + \cancel{3},\ \cancel{4},\ 6$$

$$3x + \cancel{4},\ \cancel{3},\ 2$$

(1) Multiplying across as shown by the arrows and adding, we get $29x$. This pair will not suit.

(2) Crossing these out, replace as shown. The cross product now is $27x$. This pair will not suit.

(3) Now try 6 and 2 as shown. The cross product is $28x$ and we have obtained the right pair.

$$\therefore \quad 15x^2 + 28x + 12 = (5x + 6)(3x + 2)$$

If it had been that the last pair did not suit, we would have proceeded to try 2 and 6, then 12 and 1. If these had failed we should next have tried 15 and 1 on the left side with all those in turn on the right. This would have exhausted all possible cases.

Example 2. *Find the factors of* $6x^2 + x - 15$.

It should be noted that in this case the end term of the given trinomial is negative. Hence the end terms of the required binomial factors must be opposite in sign.

Arrange one set of factors $(6x - 5)$ and $(x + 3)$ as shown in the diagram.

$$\begin{array}{llll} 6x - 5, & 5, & 15, & -15 \\ x + 3, & -3, & -1, & 1 \end{array}$$

(1) Multiplying across as shown by the arrows and adding, we get $13x$. This pair does not suit.

(2) If the other 3 pairs of factors are treated in the same way, it will be found that neither of them gives the desired result.

(3) We then try the pair of binomials $(2x + 3)$ and $(3x - 5)$ as set out below:

$$\begin{array}{ll} 2x + 3, & -3 \\ 3x - 5, & 5 \end{array}$$

Multiplying across as shown by the arrows and adding, we get $-x$, which agrees with the middle term of the trinomial except for its sign.

(4) Crossing out the end terms, $+3$ and -5 as shown, and replacing by -3 and $+5$, we obtain the desired result.

Hence $\quad \mathbf{6x^2 + x - 15 = (2x - 3)(3x + 5)}$

This method can be adopted in all cases of trinomials of the type $Ax^2 + Bx + C$, where factorisation is possible.

After a certain amount of practice in these, the student will find that in certain cases a brief inspection only may be necessary to hit upon the factors, and even in more difficult cases, the work can be considerably shortened, as much of it can be done mentally.

The important point, however, is to make sure that the factors when multiplied give the correct middle term.

D. Factors of Trinomials which Form a Perfect Square

9. In § 2 it was shown that the square of the sum of two quantities such as a and b which we express as $(a + b)^2$ is $a^2 + 2ab + b^2$.

This result consists of the sum of the squares of each quantity taken separately, together with **twice their product.**

Now for the reverse process:

Factorise	$\mathbf{4a^2 + 12ab + 9b^2}$
It is seen that	$4a^2 = (2a)^2$
and that	$9b^2 = (3b)^2$
Also	$12ab =$ **Twice** $(2a \times 3b)$

Hence,

$$\begin{aligned} 4a^2 + 12ab + 9b^2 &= (2a + 3b)(2a + 3b) \\ &= (2a + 3b)^2 \end{aligned}$$

NOTE.—**Twice** the product of the two quantities contained in either of the binomial factors must give the middle term of the original trinomial.

E. To Factorise the Difference of Two Squares

10. It has been shown that the product of the **sum** of the two quantities a and b and of their **difference** is equal to the difference of their squares.

In other words, $(a + b)(a - b) = a^2 - b^2$

Factorise $\mathbf{4m^2 - 9n^2}$

Now, $4m^2 - 9n^2 = (2m)^2 - (3n)^2$

We thus have the **difference** of the squares of $2m$ and $3n$, and conversely this difference is equal to the product of the sum $(2m + 3n)$ and of the difference $(2m - 3n)$.

$$\begin{aligned} \text{Hence} \quad 4m^2 - 9n^2 &= (2m)^2 - (3n)^2 \\ &= (2m + 3n)(2m - 3n) \\ \text{Similarly} \quad \tfrac{1}{9}x^2 - \tfrac{1}{16}y^2 &= (\tfrac{1}{3}x)^2 - (\tfrac{1}{4}y)^2 \\ &= (\tfrac{1}{3}x + \tfrac{1}{4}y)(\tfrac{1}{3}x - \tfrac{1}{4}y) \end{aligned}$$

Factorise $\mathbf{m^2 + 2mn + n^2 - 4x^2}$

The first three terms of this expression form a perfect square, *i.e.* $(m + n)^2$.

$$\begin{aligned} \therefore \quad m^2 + 2mn + n^2 - 4x^2 &= (m + n)^2 - 4x^2 \\ &= (m + n)^2 - (2x)^2 \\ &= \{(m + n) + 2x\}\{(m + n) - 2x\} \\ &= (m + n + 2x)(m + n - 2x) \end{aligned}$$

Factorise $\mathbf{9a^2 - (m - n)^2}$

$$9a^2 - (m - n)^2 = (3a)^2 - (m - n)^2$$

We thus have the difference of the squares of $3a$ and $(m - n)$.

$$\begin{aligned} \therefore \quad 9a^2 - (m - n)^2 &= \{3a + (m - n)\}\{3a - (m - n)\} \\ &= (3a + m - n)(3a - m + n) \end{aligned}$$

F. Sum and Difference of Two Cubes

11. (A)
$$\begin{aligned} (a + b)(a^2 - ab + b^2) &= a(a^2 - ab + b^2) + b(a^2 - ab + b^2) \\ &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3 = (a)^3 + (b)^3 \\ &= \text{Sum of } \textbf{cubes} \text{ of } a \text{ and } b \end{aligned}$$

(B) Similarly it can be shown that

$$\begin{aligned} (a - b)(a^2 + ab + b^2) &= a^3 - b^3 \\ &= (a)^3 - (b)^3 \\ &= \text{Difference of } \textbf{cubes} \text{ of } a \text{ and } b \end{aligned}$$

Let us take the converse of A.

$$\begin{aligned} a^3 + b^3 = (a)^3 + (b)^3 &= (a + b)(a^2 - ab + b^2) \\ &= [\text{Sum of } a \text{ and } b]\,[\text{Sq. of } a - \text{product of } a \text{ and } b + \text{Sq. of } b] \end{aligned}$$

Apply this method in factorising $8a^3 + 27x^3$

$$\begin{aligned} 8a^3 + 27x^3 &= (2a)^3 + (3x)^3 \\ &= [\text{Sum of } 2a \text{ and } 3x]\,[\text{Sq. of } 2a - \text{Product of } 2a \text{ and } 3x + \text{Sq. of } 3x] \\ &= (2a + 3x)(4a^2 - 6ax + 9x^2) \end{aligned}$$

III. FRACTIONS

12. The rules for the simplification of algebraic fractions are applied in the same way as in Arithmetic.

1. Multiplication and Division

Example. *Simplify* $\dfrac{3a - 3b}{3a + b} \times \dfrac{9a^2 - b^2}{a - b} \times (7a - 21b)$

This expression

$$\begin{aligned} &= \frac{3(a - b)}{3a + b} \times \frac{(3a + b)(3a - b)}{a - b} \times 7(a - 3b) \\ &= 3 \times 7(3a - b)(a - 3b) \text{ on omitting factors which are common to both numerator and denominator.} \\ &= 21(3a - b)(a - 3b) \end{aligned}$$

NOTE.—In every simplification, and at the outset, every numerator and denominator must be factorised when possible.

2. Addition and Subtraction of Fractions

13. Here, as in Arithmetic, we must find the lowest common denominator for all the fractions and find the

equivalents of those fractions having that common denominator.

Example 1. *Express* $\frac{a}{b} + \frac{b}{c} - \frac{c}{a}$ *as one fraction.*

The common denominator is abc.

$$\text{Then } \frac{a}{b} + \frac{b}{c} - \frac{c}{a} = \frac{a^2c}{abc} + \frac{ab^2}{abc} - \frac{bc^2}{abc}$$
$$= \frac{a^2c + ab^2 - bc^2}{abc}.$$

Example 2. *Find the value of*

$$\frac{a}{a+b} - \frac{b}{a-b} + \frac{ab}{a^2-b^2}.$$

Since $a^2 - b^2 = (a+b)(a-b)$, this is the common denominator.

Then

$$\frac{a}{a+b} - \frac{b}{a-b} + \frac{ab}{a^2-b^2} =$$
$$\frac{a(a-b) - b(a+b) + ab}{(a+b)(a-b)} = \frac{a^2 - ab - ab - b^2 + ab}{(a+b)(a-b)}$$
$$= \frac{a^2 - ab - b^2}{(a+b)(a-b)}.$$

Example 3. If $M = \frac{3a}{2m-3n} + \frac{2a}{3m-2n}$ *express* M *as a single fraction.*

$$M = \frac{3a}{2m-3n} + \frac{2a}{3m-2n}$$
$$= \frac{3a(3m-2n) + 2a(2m-3n)}{(2m-3n)(3m-2n)}$$
$$= \frac{9am - 6an + 4am - 6an}{(2m-3n)(3m-2n)}$$
$$= \frac{13am - 12an}{(2m-3n)(3m-2n)} = \frac{a(13m-12n)}{(2m-3n)(3m-2n)}.$$

Example 4. *Simplify* $\left[\frac{1+R}{R} + 1\right]\left[\frac{1-R}{R} - 1\right]$.

$$\text{Now,} \quad \frac{1+R}{R} + 1 = \frac{1+R+R}{R} = \frac{1+2R}{R}$$
$$\text{and} \quad \frac{1-R}{R} - 1 = \frac{1-R-R}{R} = \frac{1-2R}{R}.$$
$$\text{Hence the expression} \quad = \frac{1+2R}{R} \times \frac{1-2R}{R}$$
$$= \frac{1-4R^2}{R^2}.$$

EXERCISE IV

See also the miscellaneous exercises commencing on p. 87.

Section A

Find the product in each of the following cases:

1. $a(p + q - y)$.
2. $3mn(ab - cd + da)$.
3. $5x^2(3m - 2n + 5p)$.
4. $5R(R^2 - R + 1)$.
5. $\frac{1}{2}x(x^2 - 2x + 1)$.
6. $\frac{a}{3}(4a^2 - 3a + 7)$.
7. $(2a + 3b)(a + 4b)$.
8. $(x - y)(3x + 2y)$.
9. $(x - 3{\cdot}2)(x - 2{\cdot}5)$.
10. $(p + 1{\cdot}4)(2p - 3{\cdot}5)$.
11. $(1{\cdot}6 - y)(2{\cdot}5 + y)$.
12. $(5pq - mn)(2pq - 3mn)$.
13. $(7 - 8a)(5 + 4a)$.
14. $(6p^2 - q^2)(3p^2 + q^2)$.
15. $(R - 2)(2R + 3)$.

16. $(15 - 2q)(13 + 4q)$.
17. $(7ab - 8x)(5ab + 2x)$.
18. $(5p + 3q)^2$.
19. $(a - 2b)^2$.
20. $(4m + 3n)^2$.
21. $(15p - q)^2$.
22. $[(1 - p) + q][(1 + p) + 2q]$.
23. $[(3x - y) + m][(3x - y) - 3m]$.
24. $(a + b - c)(a + b + 2c)$.
25. $(R - x)(R + x)$.
26. $(2c - d)(2c + d)$.
27. $(5mn - 4pq)(5mn + 4pq)$.
28. $\left(\frac{1}{a} + x\right)\left(\frac{1}{a} - x\right)$.
29. $(2c - d)(2c + d)$.
30. $\left(\frac{3pq}{2} - \frac{4c}{5}\right)\left(\frac{3pq}{2} + \frac{4c}{5}\right)$.
31. $(x + 1{\cdot}5)(x - 1{\cdot}5)$.
32. $(2{\cdot}5a - 1{\cdot}4)(2{\cdot}5a + 1{\cdot}4)$.
33. $(p + 2q - c)^2$.
34. $(3a - 2b - 4c)^2$.

Section B

Simplify the following fractions in which the denominator is a factor of the numerator:

1. $\frac{a^2 + a - 6}{a + 3}$.
2. $\frac{x^2 + 5x + 4}{x + 4}$.
3. $\frac{4x^2 - 4px + p^2}{2x - p}$.
4. $\frac{15a^2 + ab - 6b^2}{3a + 2b}$.
5. $\frac{x^2 - 9}{x + 3}$.
6. $\frac{4m^2n^2 - 9a^2}{2mn - 3a}$.
7. $\frac{ab - ac + pb - pc}{b - c}$.
8. $\frac{6am + 4an - 9bm - 6bn}{3m + 2n}$.
9. $\dfrac{\frac{1}{x^2} - \frac{1}{a^2}}{\frac{1}{x} + \frac{1}{a}}$.
10. $\dfrac{\frac{4p^2}{9} - a^2}{\frac{2p}{3} - a}$.

Section C

Find the factors of the following:

I

1. $ax - bx + cx$.
2. $p^2q^2 - apqy + pbqy$.
3. $14a^3 - 7a^2y + 56ay^2$.
4. $54a^3b^2c - 36ab^3c^2 + 27abc^3$.
5. $\frac{ca}{x^2} - \frac{cb}{x^3} + \frac{cd}{x^4}$.
6. $\frac{10mp}{nq} - \frac{2mq}{np} + \frac{4mp}{nr}$.

II

1. $ax + bx + ay + by$.
2. $a^2c^2 - acd + abc - bd$.
3. $x^2c - x^2d - b^2d + b^2c$.
4. $2a^3 - 3a^2 + 4a - 6$.
5. $11a^3 + 55a^2 + 7a + 35$.
6. $mn(a^2 + b^2) - pq(a^2 + b^2)$.
7. $ab(x^2 + 1) - x(a^2 + b^2)$.
8. $am^2 - bm^2 + a - b$.
9. $2x^3 - x^2 + 2x - 1$.
10. $p^2 - qr - q + p^2r$.
11. $a^2 - 2ab - 3ac + 6bc$. (U.E.I.)

III

1. $a^2 + 9a + 20$.
2. $a^2 - 6a + 9$.
3. $m^2 - mn - 6n^2$.
4. $x^2 + 2x - 35$.
5. $x^2 - 5x - 14$.
6. $x^2 + x - 72$.
7. $21 + 10a + a^2$.
8. $1 - 3a + 2a^2$.
9. $p^2 + 4p - 45$.
10. $x^2 + (m + n)x + mn$.
11. $x^2 - 22xy + 85y^2$.
12. $3a^2 - 7a + 2$.
13. $4a^2 - 16a + 15$.
14. $20a^2 + 41a + 20$.
15. $9a^2 - 9a - 28$.
16. $14p^2 - 29p + 12$.
17. $12a^2 + 19ab + 5b^2$.
18. $26r^2 - 41r + 3$.

IV

1. $4a^2 - 12ab + 9b^2$.
2. $25a^2 - 60ab + 36b^2$.
3. $49m^2 + 28mn + 4n^2$.
4. $p^2 + 4p + 4$.
5. $q^2 - 8q + 16$.
6. $x^2 - x + \frac{1}{4}$.
7. $\frac{1}{a^2} + \frac{2}{ab} + \frac{1}{b^2}$.
8. $R^2 - 2R + 1$.
9. $a^2 - 9b^2$.
10. $25x^2 - 49y^2$.
11. $121x^2 - 16y^2$.
12. $\frac{1}{a^2} - \frac{1}{b^2}$.
13. $x^2 - \frac{1}{4}$.
14. $1 - \frac{81}{16}x^2$.
15. $\frac{9}{16} - 4a^2$.
16. $144p^2 - 169q^2$.
17. $a^2 - (m + n)^2$.
18. $p^2 - (2p + q)^2$.
19. $(p - q)^2 - r^2$.
20. $(a + b)^2 - a^2$.
21. $a^2 + 2ab + b^2 - c^2$.
22. $m^2 - n^2 - 2np - p^2$.
23. $x^2 - a^2 + 2ab - b^2$.

V

1. $m^3 - 27n^3$.
2. $8a^3 - 64b^3$.
3. $\frac{1}{m^3} + \frac{1}{n^3}$.
4. $R^3 + 1$.
5. $2a^2 + 14a + 24$.
6. $15ax^2 - 35ax + 10a$.
7. $20 + 36x - 8x^2$.
8. $48mx^2 - 24mx - 45m$.

Section D

Simplify the following fractions by first factorising where possible:

1. $\dfrac{a^2 - b^2}{a^2 + 2ab + b^2} \times \dfrac{ab + b^2}{a^2 - ab}$.
2. $\dfrac{a^2 - 49}{a^2 - 9} \div \dfrac{a + 7}{a + 3}$.
3. $\dfrac{p^4 - q^4}{p^2 - 2pq + q^2} \div \dfrac{p^2 + q^2}{q(p - q)}$.
4. $\dfrac{6a^2 + 5a + 1}{6a^2 - a - 1} \times \dfrac{2a^2 - 11a + 5}{2a^2 - 11a - 6}$.
5. $\dfrac{m^4 - 27m}{m^2 - 9} \div \dfrac{m^2 + 3m + 9}{m + 3}$.

Section E

Express the following in their simplest form:

1. $\dfrac{a}{x} + \dfrac{a}{3x} - \dfrac{2a}{4x}$.
2. $\dfrac{m}{nq} + \dfrac{n}{mq} + \dfrac{q}{mn}$.
3. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$.
4. $\dfrac{a}{a + b} - \dfrac{a}{a - b}$.
5. $\dfrac{x + 3}{x - 3} - \dfrac{x - 3}{x + 3}$.
6. $\dfrac{3}{1 - a} + \dfrac{4}{(1 - a)^2}$.
7. $\dfrac{a - b}{c - d} - \dfrac{b - a}{d - c}$.
8. $\dfrac{1}{a - 2b} - \dfrac{a^2 + 4b^2}{a^3 - 8b^3}$.
9. $\dfrac{1}{p + q} + \dfrac{1}{p - q} + \dfrac{2p}{p^2 - q^2}$.
10. $\dfrac{1}{a^2 - 5a + 4} - \dfrac{1}{a^2 - 4a + 3}$.

Section F. Miscellaneous Exercises

1. Factorise (1) $36x^2 - 81y^2$.
(2) $4x^2 + 5xy - 6y^2$.
(3) $a^3b - 3a^2b + 2ab$.

(U.L.C.I.)

2. Write down each of the following and fill in the blanks:

(1) $(7{\cdot}4 \times 13^2) + (7{\cdot}4 \times a^2) = 7{\cdot}4\ (\qquad)$.

(2) $\dfrac{2x - y}{x - y} = \dfrac{2(y - 2x)}{(\qquad)}$.

(3) $3a^2 + 5ab - 2b^2 = (3a^2 + 6ab) - (ab + \quad)$,
$= 3a(\qquad) - b(\qquad)$,
$= (\qquad)(a + 2b)$.

(N.C.T.E.C.)

3. The cast-iron base of a machine has the shape of an inverted open box a in. long, b in. wide, c in. high overall, the metal being t in. thick. Calculate the volume of metal in the casting in two ways: (i) by subtracting the volume of the inner open space from the whole volume occupied; and (ii) by computing and adding the volumes of the walls and top. Show that the two expressions when simplified are the same.

If $a = 36$, $b = 18$, $c = 6$ and $t = \frac{5}{8}$, find the volume of metal in the casting, and its weight if cast iron weighs 0·26 lb per cu in.

4. Write down the algebraic expressions which indicate:

(1) The sum of the squares of two numbers indicated by a and b.

(2) The square of the sum of two numbers indicated by p and q.

(3) The fraction obtained by dividing 3 by the sum of five and the square of a number indicated by x.

(N.C.T.E.C.)

5. Express in its simplest form each of the expressions

(1) $[(x + y)^2 - (x^2 + y^2)]$.

(2) $\dfrac{1}{q}[(h - \mathrm{K})(p - q) - hp + \mathrm{K}p]$.

(N.C.T.E.C.)

6. Express in their simplest forms each of the expressions

(1) $[(x + y)(u + v) - (yu + yv)] \div x$.

(2) $[(r + s)(r - t) + st] \div r$.

(N.C.T.E.C.)

7. Multiply $a^2 - 2ax + 4x^2$ by $a^2 + 2ax + 4x^2$.

(Cannock.)

8. Simplify the following, and express each result as a single fraction:

(a) $\dfrac{4}{x - 4} - \dfrac{16 + 3x}{x^2 - 16}$.

(b) $\dfrac{x^2 - 5x + 6}{x^2 - 16} \times \dfrac{x^2 + 5x + 4}{x^2 - 4} \div \dfrac{x - 3}{x - 4}$.

(Cheltenham.)

9. Factorise (i) $4x^2 + 12ab - 9a^2 + y^2 + 4xy - 4b^2$.
(ii) $(2a + 3b - c)^2 - (a - 2b + c)^2$.

(Handsworth.)

10. Factorise: (a) $3x^2 - 7x - 6$.
(b) $16a^2 - 49b^4$.
(c) $ab - 2b + 3a - 6$.

(Sunderland.)

11. (i) Reduce the expression $\dfrac{3a^{\frac{2}{3}}b^{\frac{3}{4}}}{a^{\frac{1}{2}}b^{\frac{1}{4}}} \div \dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}$ to its simplest terms and evaluate it when $a = 27$, $b = 16$.

(ii) Factorise the expressions: $(2b + a)^2 - a^2$; $x^2 - 3x - 10$; $ax^2 + bx^2 - ay^2 - by^2$.

(Sunderland.)

12. (a) Simplify the expression

$$\frac{3}{2x - 5} - \frac{2}{2x + 1}.$$

(b) Factorise (i) $9a^2 - 16b^2$, (ii) $2x^2 - x - 21$.

(c) If $x = a + 1$ and $y = a^2 + a$, express y in terms of x only. For what values of a are x and y equal?

(U.L.C.I.)

13. (*a*) Factorise (i) $4x^2 - \frac{9}{16x^2}$.

(ii) $ab + ac - 2b - 2c$.

(*b*) (i) Simplify $\frac{a^2 + 2ab}{a^2 - 2ab - 3b^2} \div \frac{a^2 - 4b^2}{a^2 - ab - 2b^2}$.

(ii) Find, in its simplest form in terms of x, the value of $y - \frac{1}{y}$, when $y = \frac{x-1}{x+1}$.

(Surrey County Council.)

14. (i) Simplify

$$\left(\frac{p^2}{q^3}\right)^2 \times \frac{q^4}{r^5} \times \frac{r^6}{p^3}.$$

(ii) Simplify

$$\frac{(x^2 - 9y^2)(x^2 - 4y^2)}{x^2 + xy - 6y^2},$$

giving the answer without brackets.

(iii) Express as a single fraction $1 - \frac{1}{a} - \frac{b-1}{b}$.

(Nuneaton.)

15. Find the value of $\frac{1}{R}$ if $\frac{1}{R} = \frac{1}{r+s} - \frac{1}{r-s}$.

16. (*a*) Find the value of $\frac{1}{P}$ if $\frac{1}{P} = \frac{2}{p-q} + \frac{3}{p+q}$.

(*b*) What is the value of 5P.

17. Simplify $(3a - x)(a + 2x) - \{(a - x)^2 + 2ax\}$.
Express your result in factors. (U.L.C.I.)

18. Factorise the expressions

(1) $\frac{\pi D^4}{4} - \frac{\pi d^4}{4}$. (2) $6m^2 + 19m + 15$.

(U.L.C.I.)

19. Evaluate with as little labour as possible

$$\frac{8(23{\cdot}7)^2 - 10(23{\cdot}7)(45{\cdot}4) + 3(45{\cdot}4)^2}{4(23{\cdot}7) - 3(45{\cdot}4)}.$$

(N.C.T.E.C.)

20. Find the difference between $\frac{f}{(a-b)^2}$ and $\frac{f}{(a+b)^2}$.

State what this difference would approximate to if b was so small compared with a that terms including b to the second or higher power may be neglected. (U.E.I.)

21. Simplify

(1) $\frac{a^2 + 2ab + b^2}{a - b} \times \frac{a^2 - b^2}{(a+b)^2}$.

(2) $\frac{ax^2 + 2a^2x + a^3}{x^3 + 2ax^2 + a^2x}$.

(U.E.I.)

CHAPTER 5

EQUATIONS

1. In Chapter 3 it was shown how all the operations of arithmetic could still be expressed when letters were used in place of numbers to represent the magnitudes of quantities of any kind. Thus we could speak of m pence, l tons, y persons; and use these letters in algebraic expressions as freely as if we knew the actual numbers for which the letter symbols stood. We now proceed to use symbols for the magnitudes of quantities concerned in problems, and to incorporate them in statements relating to the problems. This we can do just as readily as we could actual numbers. In turn these symbolic statements may be used to show what values the symbols must have if the conditions of the problem are to be satisfied.

Example 1. Suppose the quantity p shillings is taken to represent the subscription to a certain society, and that at four different centres the number of subscribers is respectively 54, 76, 32 and 48.

The amounts subscribed by these centres are $54p$, $76p$, $32p$ and $48p$ shillings.

If, further, we know that the sum raised is 420 shillings, we can say that

$$54p + 76p + 32p + 48p = 420 \text{ shillings}$$

that is,

$$210p = 420$$

Evidently, then, the subscription p must be 2 shillings,

$$\text{or} \quad p = \tfrac{420}{210} = 2 \text{ shillings.}$$

The important point to notice is that the amount of the subscription is given in two different forms:

(1) In the form $54p + 76p + 32p + 48p$;
(2) as 420 shillings.

Since these must be equal, this may be expressed by writing down:

$$54p + 76p + 32p + 48p = 420$$

Such a statement of equality is called an Equation, and when there is no higher power than the first of the unknown quantity, it is called a **Simple Equation,** or an equation of the first degree. As we deal with a series of *simple* equations we shall find that only one particular value assigned to the symbol will make the equation true.

Example 2. *A man buys 24 packets of envelopes for office use, each packet containing an unknown number m of envelopes.*

On two successive days he distributes 10 and 8 packets respectively and finds later that 150 envelopes are left.

How many were there in each packet?

Original number of envelopes $= 24m$.
Numbers distributed were $10m$ and $8m$.
$\therefore$ The number remaining is $24m - 8m - 10m$.
But the number remaining is known to be 150.
Hence we have the equation:

$$24m - 10m - 8m = 150$$

that is,

$$6m = 150$$

Dividing by the coefficient of m, we find that

$$m = 25.$$

and this is the number of envelopes in a packet.

2. This process of simplifying the equation and finding the value of the unknown is termed **"Solving the Equation."**

This value is called a **root** of the equation and is said to satisfy the equation. The reason of this is, that if it be substituted in the original equation, both sides should be

equal, *i.e.* the equation is satisfied by this value. In this way the accuracy of the result may be tested.

In general, equations require far more simplification than in those shown above before the value of the unknown can be determined, and hence it is necessary to call to our aid certain truths termed **axioms.**

I. If equal quantities be added to two quantities that are already equal, the results will be equal.

II. If equal quantities be subtracted from two quantities that are already equal, the remainders will be equal.

III. Equal quantities, when multiplied or divided by the same quantity, will give results that are equal.

Example 3. *Solve the equation* $8x - 2x + 3 = 3x + 12$.

Our problem is to find that value of x, and there is only one, which will satisfy the equation—that is, make the above statement true.

The first step is to get all the terms containing x on the **left-hand side** (L.H.S.) and other quantities on the **right-hand side** (R.H.S.).

Applying the axioms set out above, we will subtract $3x$ and 3 from each side of the equation.

This gives us

$$8x - 2x + 3 - 3 - 3x = 3x - 3x + 12 - 3$$

that is, $$8x - 2x - 3x = 12 - 3$$

Comparing this with the original equation, we see that the same result could have been obtained by transferring the $+ 3$ from the L.H.S. to R.H.S. and changing its sign, and also transferring the $+ 3x$ from the R.H.S. to the L.H.S. and changing its sign.

If this be done we are, in effect, using the axioms stated above.

In future examples quantities will be transferred in this way simply by means of a change of sign.

Finally
$$\begin{aligned} 8x - 5x &= 9 \\ 3x &= 9 \\ \therefore\ x &= 3 \end{aligned}$$

Verification.

Substitute for x in the original equation

$$\begin{aligned} 8x - 2x + 3 &= 3x + 12 \\ \text{L.H.S.} &= 8x - 2x + 3 \\ &= 24 - 6 + 3 \\ &= 21 \\ \text{R.H.S.} &= 9 + 12 \\ &= 21 \end{aligned}$$

Hence the L.H.S. does equal the R.H.S. provided $x = 3$.

Example 4. *Solve* $\frac{5}{2}x - 7 = \frac{4}{3}x + 12$

$$\frac{5}{2}x - \frac{4}{3}x = 12 + 7 \qquad \text{(Axiom II)}$$

that is, $$\frac{7}{6}x = 19$$

$$\therefore\ x = \frac{19}{\frac{7}{6}} = \frac{114}{7} = 16\tfrac{2}{7} \qquad \text{(Axiom III)}$$

Example 5. *Solve* $2(x - 5) - 3(x + 7) = x + 12$.

The first step here is to clear the equation of brackets, and to note in doing so that a **minus** sign before the bracket indicates a change of sign for each term within the bracket.

Then

$$2x - 10 - 3x - 21 = x + 12$$

$$2x - 3x - x = 12 + 10 + 21 \qquad \text{(Axioms I and II)}$$

that is, $$-2x = 43$$

or $$2x = -43 \qquad \text{(Axiom III)}$$

$$\therefore\ x = -21\tfrac{1}{2}$$

Example 6. *Find the value of x which makes* $\frac{15}{x} = \frac{3}{4}$.

Here we multiply both sides by the common denominator $4x$ in order to clear the equation of fractions.

Then $\quad \frac{15}{x} \times 4x = \frac{3}{4} \times 4x \qquad$ (Axiom III)

Cancelling, $\quad 60 = 3x$

Then $\quad x = 20$

Example 7. *Solve* $\frac{5}{x-3} = \frac{8}{x+9}$.

Then

$$\begin{aligned} 5(x+9) &= 8(x-3) && \text{(Axiom III)} \\ 5x + 45 &= 8x - 24 \\ 5x - 8x &= -45 - 24 && \text{(Axiom II)} \\ -3x &= -69 \\ 3x &= 69 \\ \therefore \quad x &= 23 \end{aligned}$$

Example 8. *Find* W *from the formula* $R = W\left(\frac{a + 6t}{a}\right)$ *if* $R = 18$, $a = 2{\cdot}8$ *and* $t = 1{\cdot}2$.

Substituting with the given values, we obtain the result

$$18 = W\frac{(2{\cdot}8 + 7{\cdot}2)}{2{\cdot}8}$$

an equation with W as the unknown.

This by multiplying both sides by 2·8 becomes

$$18 \times 2{\cdot}8 = W(2{\cdot}8 + 7{\cdot}2) \qquad \text{(Axiom III)}$$

that is, $\quad 50{\cdot}4 = 10W$

$\therefore \quad W = 5{\cdot}04$

Example 9. *A velocity of* V *ft per sec is the same as* $\frac{1}{3}(5V - 45)$ *m.p.h. Find the value of* V.

Miles per hour can be converted into ft per sec by multiplying by $\frac{22}{15}$ (see Chapter 6, p. 116).

Then $\frac{1}{3}(5V - 45)$ m.p.h. is the same velocity as $\frac{1}{3}(5V - 45)\frac{22}{15}$ ft per sec.

Hence the velocity is expressed in ft per sec in two ways:

$$V \text{ and } \tfrac{1}{3}(5V - 45)\tfrac{22}{15}$$

Equate these and solve for V.

Then $\quad V = \frac{1}{3}(5V + 45)\frac{22}{15}$

that is, $\quad V = \frac{22}{45}(5V - 45)$

Multiplying both sides by 45, the common denominator, we have:

$$45V = 22(5V - 45) \qquad \text{(Axiom III)}$$

that is, $\quad 45V = 110V - 990$

$$990 = 65V \qquad \text{(Axioms I and II)}$$

$$\therefore \quad V = \tfrac{990}{65} = \tfrac{198}{13} = 15\tfrac{3}{13} \text{ ft per sec.}$$

Problems Involving Simple Equations

3. The examples worked above illustrate the methods which can be employed in solving a Simple Equation when the equation is given.

The importance of equations, however, really lies in their application to the solution of Problems, and in such cases it is necessary, first, to form equations which are consistent with the data provided by those problems.

Example 1. *A rectangular metal plate is 25 cm long. A strip 4·5 cm wide is cut off from one end, and a second strip 1·15 cm wide is cut off from the other.*

The remainder weighs 139·32 gm. Find the width of the plate if 1 sq cm of it weighs 0·9 gm.

Let x cm represent the width of the plate.

Then its area is $25x$ sq cm.

Areas of strips cut off are $4{\cdot}5x$ sq cm and $1{\cdot}15x$ sq cm.

Then area of remainder $= (25x - 4{\cdot}5x - 1{\cdot}15x)$ sq cm.

The weight of this is $0{\cdot}9(25x - 4{\cdot}5x - 1{\cdot}15x)$ gm.

But the weight of the remainder is given as 139·32 gm.

$$\therefore \quad 0{\cdot}9(25x - 4{\cdot}5x - 1{\cdot}15x) = 139{\cdot}32$$

that is $\quad 25x - 4{\cdot}5x - 1{\cdot}15x = 154{\cdot}8 \quad$ (Axiom III)

$$25x - 5{\cdot}65 = 154{\cdot}8$$
$$19{\cdot}35x = 154{\cdot}8$$
$$\therefore \quad x = 8 \text{ cm.}$$

Example 2. *What weight of tin must be melted up with 48 lb of copper in order that the alloy may contain 16·5% of tin?*

Let x lb represent the weight of tin added.
Then the weight of the alloy is $(48 + x)$ lb.
The tin has to represent 16·5% of this,

that is, $\quad \dfrac{16{\cdot}5}{100}$ of $(48 + x)$ or $\dfrac{16{\cdot}5}{100}(48 + x)$ lb.

But the weight of the tin in the alloy is x lb.

$$\therefore \quad x = \frac{16{\cdot}5}{100}(48 + x)$$

that is, $\quad 100x = 16{\cdot}5(48 + x) \quad$ (Axiom III)

$$100x = 792 + 16{\cdot}5x$$
$$83{\cdot}5x = 792$$
$$\therefore \quad x = \frac{792}{83{\cdot}5} = 9{\cdot}485 \text{ lb.}$$

Example 3. Example 13 on p. 43 can be restated as a problem leading to a simple equation for the height H.

What is the height H *in. of a flat-bottomed cup of diameter* 4 *in. which can be made from a circular blank of diameter* 12 *in., if the thickness of the bottom and sides of the cup remains the same as the thickness of the blank?* Remember that the circumference of a circle is given by the formula $2\pi r$, and the area by the formula πr^2.

Since the thickness in the cup remains the same as the thickness of the blank, the combined area of the bottom

and sides of the cup must be the same as the area of the blank.

The area of the cup bottom is $\pi r^2 = 4\pi$ sq in.
The area of the cup sides is $2\pi r\text{H} = 4\pi\text{H}$ sq in.
The total area in the cup is $4\pi + 4\pi\text{H}$
$$= 4\pi(\text{H} + 1) \text{ sq in.}$$

But the area of the blank is $\pi\,\text{R}^2$, where $\text{R} = 6$,

that is, $\quad 36\pi$ sq in.

Thus $\quad 4\pi(\text{H} + 1) = 36\pi$

$$\text{H} + 1 = 9,$$

and $\quad \text{H} = 8$

The height of the cup is therefore 8 in. The very simple arithmetic involved in the solution of this equation arises from the values chosen for the two diameters. Numbers chosen at random would work out quite differently. Readers should make a clear dimensioned sketch of the cup and blank.

Equations with Two Unknowns

4. It frequently happens that the solution of a problem involves the use of more than one unknown.

We now consider cases which involve two unknowns.

Suppose that 3 times a certain number added to twice a second number gives 24 as a result.

Let N represent the first number and n represent the second number.

Then $\quad 3\text{N} + 2n = 24$

If we endeavour to find the value of N in the usual way

$$3\text{N} = 24 - 2n$$

and $\quad \text{N} = \dfrac{24 - 2n}{3}$

This does not give the actual numerical value of N, but a result which involves the other unknown n.

If we knew the value of n we could find the value of N by substituting for n in the fraction.

For example, let $n = 3$.

Then $\quad N = \dfrac{24 - 6}{3} = 6$

If $n = 5$, $\quad N = \dfrac{24 - 10}{3} = 4\frac{2}{3}$

If $n = 4\frac{1}{2}$, $\quad N = \dfrac{24 - 9}{3} = 5$

This method could be continued indefinitely, so that for every value of n there is a corresponding value of N, and there is apparently an endless number of possibilities.

Now, the original statement presupposes only **one pair** of values for N and n, so evidently some further information must be available in order that the right pair can be found.

This additional information will permit a second equation to be set down. The new facts might be that **3 times the first number added to the second number gives 21 as a result,** so that—

$$\mathbf{3N + n = 21}$$

If now we take the value of N found in the *first* equation, viz. $N = \dfrac{24 - 2n}{3}$, and substitute this for N in the *second* equation, we have:

$$3\left(\frac{24 - 2n}{3}\right) + n = 21$$

a simple equation involving one unknown;

that is,
$$\begin{aligned} 24 - 2n + n &= 21 \\ -n &= 21 - 24 \\ -n &= -3 \\ \therefore \quad n &= 3 \end{aligned}$$

Substituting $n = 3$ in the equation $3N + n = 21$,

we have
$$\begin{aligned} 3N + 3 &= 21 \\ 3N &= 18 \\ \therefore \quad N &= 6 \end{aligned}$$

Thus $n = 3$ and $N = 6$ is the pair of values which satisfies both equations.

Alternative Method

5. These values for N and n can also be determined as follows.

Rewriting the equations we have:

$$3N + 2n = 24 \quad . \quad . \quad . \quad . \quad (1)$$
$$3N + n = 21 \quad . \quad . \quad . \quad . \quad (2)$$

The difference between the two L.H. sides is n.
The difference between the two R.H. sides is 3.
These differences must be equal.
Hence $n = 3$

Then from (2)
$$\begin{aligned} 3N + 3 &= 21 \\ 3N &= 18 \\ \therefore \quad N &= 6 \text{ as before.} \end{aligned}$$

These values of N and n satisfy *both* equations.

It will be noted that in paragraph 4, by making substitutions in the *first* equation, we found that

when $n = 5$, $N = 4\frac{2}{3}$
when $n = 4\frac{1}{2}$, $N = 5$

These values satisfy the **first** equation but not the **second.**

If N and n have definite values, any equation involving them must be satisfied by those values.

We thus see that when two unknowns have to be found, we require two equations involving them.

Further, *three unknowns* in a problem would require *three equations* for their determination.

Solution of Equations Involving Two Unknowns

1st Method. Substitution

6. In paragraph 4, when dealing with the unknowns N and n, we first found the value of N in terms of n from the first equation.

This value of N was then substituted in the second equation, the result being that we obtained a simple equation involving n only, from which the numerical value of n was determined.

Knowing n and substituting its value in either of the two given equations involving N and n, we obtain an equation involving N only which is solved in the usual manner.

This is known as the **Substitution Method,** further examples of which are given below.

Example 1. *Solve* (1) $5x - 3y = -37$

(2) $2x + 3y = 2$

From (1) $\quad 5x = 3y - 37$

that is, $\quad x = \dfrac{3y - 37}{5}$

Substituting this value of x in (2) we have:

$$2\left(\frac{3y-37}{5}\right) + 3y = 2$$

that is,
$$2(3y - 37) + 15y = 10 \qquad \text{(Axiom III)}$$
$$6y - 74 + 15y = 10$$
$$21y = 84$$
$$\therefore \; y = 4$$

Substituting for y in (1) we have:

$$5x - 12 = -37$$
$$5x = -25$$
$$\therefore \; x = -5$$

Example 2. *Find the values of* R_1 *and* R_2 *which will satisfy the equations*

(1) $0{\cdot}5R_1 + 1{\cdot}2R_2 = 1{\cdot}486$

(2) $4{\cdot}5R_1 - 2R_2 = 4{\cdot}67$

From (1) $\quad 0{\cdot}5R_1 = 1{\cdot}486 - 1{\cdot}2R_2$

that is, $\quad R_1 = 2{\cdot}972 - 2{\cdot}4R_2 \qquad$ (Axiom III)

Substituting for R_1 in (2) we have:

$$4{\cdot}5(2{\cdot}972 - 2{\cdot}4R_2) - 2R_2 = 4{\cdot}67$$
$$13{\cdot}374 - 10{\cdot}8R_2 - 2R_2 = 4{\cdot}67$$
$$-12{\cdot}8R_2 = -8{\cdot}704$$
$$\therefore \; R_2 = 0{\cdot}68$$

Substituting for R_2 in (2) we have:

$$4{\cdot}5R_1 - 1{\cdot}36 = 4{\cdot}67$$

that is,
$$4{\cdot}5R_1 = 6{\cdot}03$$
$$\therefore \; R_1 = 1{\cdot}34.$$

2nd Method. Elimination

7. It will be found that in certain cases of Simultaneous Equations, the **Substitution Method** is unnecessarily cumbersome, so that, where possible, the method of **Elimination** should be employed in order to shorten the working.

This method was the one employed as an alternative in paragraph 5 in dealing with the unknown quantities N and n.

Other examples are given below.

Example 1. *Solve* (1) $3x + 2y = 12$

(2) $x + 3y = 11$

By multiplying equation (2) by 3 throughout we have the coefficient of x the same in both equations.

Thus
$$3x + 2y = 12 \quad . \quad . \quad . \quad (1)$$
$$3x + 9y = 33 \quad . \quad . \quad . \quad (2) \qquad \text{(Axiom III)}$$

Subtracting to eliminate x we have:

$$-7y = -21$$

Then $\quad y = 3$

Substituting for y in (1):

$$3x + 6 = 12$$
$$3x = 6$$
$$\therefore \quad x = 2$$

Example 2. *What are the values of x and y which will satisfy the equations*

$$(1) \quad \frac{x}{7} - \frac{y}{2} = -3$$
$$(2) \quad \frac{x}{3} + \frac{y}{4} = 10$$

In this example we must clear each equation of fractions by multiplying throughout by its own common denominator. We then have:

$$\left(\frac{x}{7} \times 14\right) - \left(\frac{y}{2} \times 14\right) = -3 \times 14 \quad . \quad (1)$$
(Axiom III)
$$\left(\frac{x}{3} \times 12\right) + \left(\frac{y}{4} \times 12\right) = 10 \times 12. \quad . \quad (2)$$
(Axiom III)

that is, $\quad 2x - 7y = -42 \quad . \quad . \quad . \quad . \quad (1)$
and $\quad 4x + 3y = 120 \quad . \quad . \quad . \quad . \quad (2)$

To eliminate x, multiply No. (1) by 2.

Then $\quad 4x - 14y = -84$
$\quad 4x + 3y = 120$
Subtracting $\quad -17y = -204$
$\quad 17y = 204$
Then $\quad y = 12$

Substitute in any one of the equations above, preferably the one involving small quantities.

Then $\quad 2x - 7y = -42$
$\quad 2x - 84 = -42$
$\quad 2x = 42$
$\quad \therefore \quad x = 21$

Example 3. *Solve the equation*

$$(1) \quad \frac{1}{R_1} - \frac{1}{R_2} = \frac{4}{35}$$
$$(2) \quad \frac{1}{R_1} + \frac{2}{R_2} = \frac{6}{7}$$

In this equation, since the unknowns are expressed in terms of their reciprocals, we solve for $\frac{1}{R_1}$ and $\frac{1}{R_2}$.

Eliminate $\frac{1}{R_1}$ by subtracting (2) from (1).

This gives us $\quad -\frac{3}{R_2} = \frac{4}{35} - \frac{6}{7}$

or $\quad \frac{3}{R_2} = \frac{26}{35}$

Then $\quad \frac{1}{R_2} = \frac{26}{105}$

$\therefore \quad R_2 = \frac{105}{26} = 4\frac{1}{26}$

Substituting in (1):

$$\frac{1}{R_1} - \frac{26}{105} = \frac{4}{35}$$
$$\frac{1}{R_1} = \frac{4}{35} + \frac{26}{105} = \frac{38}{105}$$

Hence $\quad R_1 = \frac{105}{38} = 2\frac{29}{38}$

Example 4. *The relation between* R *and* t *is given by the equation* $R = at + b$.

Find the values of a *and* b *if* $R = 11{\cdot}5$ *when* $t = 20$ *and* $R = 13{\cdot}1$ *when* $t = 60$. (U.L.C.I.)

If the pairs of values of R and t are substituted in the given relation, we shall obtain two equations involving a and b.

Thus
$$(1)\quad 11{\cdot}5 = 20a + b$$
$$(2)\quad 13{\cdot}1 = 60a + b$$

Subtracting (2) from (1):

$$-1{\cdot}6 = -40a$$

that is,
$$40a = 1{\cdot}6$$
$$\therefore\quad a = 0{\cdot}04$$

Substituting for a in (1) we have:

$$11{\cdot}5 = (20 \times 0{\cdot}04) + b$$
$$11{\cdot}5 = 0{\cdot}8 + b$$
$$\therefore\quad b = 10{\cdot}7$$

$\therefore$ the relation is $R = 0{\cdot}04t + 10{\cdot}7$

Problems Involving Simultaneous Equations

8. It should now be clear to the student that if a problem involves two unknowns, it is necessary, first, to build up two equations connecting them from the data which the question provides, and then to proceed as usual to solve these simultaneous equations.

Example 1. *An alloy containing* 7 *cc of copper and* 5 *cc of tin weighs* 98·8 *gm, while another alloy containing* 4·5 *cc of copper and* 3·5 *cc of tin weighs* 65·6 *gm. Find the weight of* 1 *cc of copper and* 1 *cc of tin.*

Let $W \equiv$ the weight of 1 cc of copper.
Let $w \equiv$ the weight of 1 cc of tin.
Then $7W + 5w \equiv$ the weight of the first alloy.

Hence $\quad 7W + 5w = 98{\cdot}8 \quad . \; . \; . \; . \; (1)$
Similarly $\quad 4{\cdot}5W + 3{\cdot}5w = 65{\cdot}6 \quad . \; . \; . \; . \; (2)$

Multiply (1) by 3·5 and (2) by 5.
We then have:

$$24{\cdot}5W + 17{\cdot}5w = 345{\cdot}8 \quad . \; . \; . \; (1)$$

and
$$22{\cdot}5W + 17{\cdot}5w = 328 \quad . \; . \; . \; (2)$$

Subtracting (2) from (1):

$$2W = 17{\cdot}8$$
$$\therefore\quad W = 8{\cdot}9 \text{ gm}$$

Substituting for W in (1) above:

$$62{\cdot}3 + 5w = 98{\cdot}8$$
$$5w = 36{\cdot}5$$
$$\therefore\quad w = 7{\cdot}3 \text{ gm}$$

Example 2. In Mechanics it is often necessary to find the supporting forces at the two ends of a beam carrying specified loads. This problem involves two unknowns (the two reactions) which appear in two simple equations. It is necessary to make use of the well-known fact that the turning effect, or moment, of a force about an axis is found by multiplying the force by its perpendicular distance from the axis.

A plank rests upon two walls 11 *ft apart. A builders' labourer wheeling a barrow stands on the plank. His weight and that of the barrow load apply two vertical forces to the plank as indicated in the figure. What are the two supporting forces?*

Let us call the supporting forces at ends A and B respectively R_A lb and R_B lb (using the letter R because it is the first letter of the word Reaction).

Now the plank rests securely upon its two supports. There is no turning about any axis at all. Whatever axis we choose, if we write down the turning moments about it,

they will add up to nothing. Let us choose O, the centre of the plank, as the point where a supposed axis of rotation intersects the figure. For this example let us neglect the weight of the plank, and let us write down the moments about the axis through O of all the vertical forces applied to the plank. If we measure forces in pounds weight, and distances along the plank in feet, all our moment products

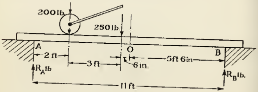


FIG. 21.

will be numbers of pound-feet. Let us regard the moments as positive, or +, if they tend to turn the plank "clockwise." The total moment must be nothing. Then, from Fig. 21,

The total moment about O = 0

$$= R_A \text{ lb} \times 5\tfrac{1}{2} \text{ ft} - 200 \text{ lb} \times 3\tfrac{1}{2} \text{ ft} - 250 \text{ lb} \times \tfrac{1}{2} \text{ ft} - R_B \text{ lb} \times 5\tfrac{1}{2} \text{ ft}$$

all terms being in lb ft.

That is:
$$(R_A - R_B) \times 5\tfrac{1}{2} = 700 + 125$$
$$= 825$$
$$R_A - R_B = \frac{1650}{11} = 150 \quad . \quad . \quad \text{(i)}$$

But R_A and R_B will add up to 450 lb, the total of the applied loads; so

$$R_A + R_B = 450 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Adding the corresponding sides of the two equations—

$$2R_A = 600$$
$$R_A = 300, \quad \text{and from (ii)}$$
$$R_B = 150.$$

The above is the systematic method by forming two equations in R_A and R_B. But R_B could be eliminated from the beginning by choosing an axis through B for our moments.

Thus $\quad 11R_A = 9 \times 200 + 6 \times 250$

or $\quad R_A = \dfrac{3300}{11} = 300$

EXERCISE V

SECTION A

Simple Equations

1. Find x when $6x = 4{\cdot}5x + 18$.
2. Solve $7x + 10 = 4x + 19$.
3. Solve $5(3x - 4) = 40$.
4. Solve $3x + 5 = x + (3x - 12)$.
5. For what value of n is $3n + 7$ equal to $14 + 2{\cdot}5n$?
6. Solve for r, $12r - 5(r - 1) = 2r + 6$.
7. Solve (a) $4(x + 2) - 3(4 - x) + 24 = 34$.
   (b) $5(x + 2) - 3(x - 3) = 23$.
   (c) $3(x - 1) - 4(2 - 3x) = 19$. (U.L.C.I.)
8. Solve $(1 - x) - 3(x - 4) = 33$.
9. Solve for t, $2t - 4 = 3(t - 1{\cdot}6)$.
10. Solve for n, $2n = 0{\cdot}58(12 - n)$.
11. Solve for n, $3(n - 7) = 6 - 4(3 - n)$.
12. For what value of r is $18{\cdot}4$ equal to $2(3{\cdot}5r - 1)$?
13. Find n when $15{\cdot}8 = \dfrac{56}{3n}$.
14. Find n when $\dfrac{7{\cdot}5}{n} = \dfrac{5}{2}$.
15. Find c if $\dfrac{18}{2c} = 3{\cdot}8$.

16. If $C = \frac{V}{R}$, (a) find V when C = 8, R = 4·5,
(b) find R when C = 7·5, V = 60.

17. For what value of x is $3(x - 5)$ equal to $\frac{4x + 3}{2}$?

Solve the following equations:

18. $\frac{x}{3} - \frac{x}{4} = \frac{1 - x}{6}$.

19. $\frac{3 - x}{4} = \frac{x}{6}$.

20. $n - 7 = \frac{2n - 5}{6}$.

21. $\frac{x - 1}{x - 2} = 3$.

22. $\frac{1 - r}{r + 1} = 4$.

23. $2 = \frac{100 - t}{360 - t}$.

24. $0·8 = \frac{1·5n}{4 + 1·4n}$.

25. Solve for x, $\frac{1}{5}(x + 3) - \frac{1}{2}(x + 2) = \frac{1}{4}(x + 8)$.

26. Solve for p, $\frac{3p + 6}{5} - \frac{4p + 3}{4} = \frac{2p + 7}{6}$.

27. The three angles of a triangle are given in the form $x°$, $(x + 3)°$, $(x - 9)°$. If the sum of these be 180°, find the three angles.

28. The perimeter of a rectangle is 44 in. If one of two adjacent sides be 1·8 in. longer than the other, what are the lengths of the sides?

29. In the formula $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$, if $v = \frac{4}{5}u$ and $f = 8$, find u.

30. If $\frac{p^2 - 2ap}{4} = \frac{3a + 7}{6}$ when $p = 3$, find a.

31. A rectangular box with square ends has its length 10 in. greater than its breadth and the total length of its edges is 152 in. What is its width?

32. If $R_2 = R_1(24 - at + bt^2)$ find the value of a when $t = 1·6$, $b = 3$, $R_1 = 17$ and $R_2 = 26$.

33. Determine L from the equation:

$$40L + 40(100 - 80) = (354·4 + 121·4 \times 0·095)(80 - 20).$$
(U.L.C.I.)

34. Find the value of $\frac{1}{x}$ given that

$$\frac{5}{2x + 5} = \frac{4}{x + 5}.$$ (N.C.T.E.C.)

35. Solve the equation $\frac{3p + 23}{3p + 12} = \frac{4}{3}$. (N.C.T.E.C.)

36. Find the value of R from the following equation:

$$(R - 3)(2R + 6) = 2R(R - 18).$$ (U.L.C.I.)

37. Two cars A and B are travelling on a road which runs east and west, at such speeds that at any instant, t minutes past noon, A is $(30t - 220)$ yd east of B. At what instant is (1) A 110 yd east of B, (2) A 100 yd west of B? (N.C.T.E.C.)

38. "Four times the sum of a certain number and five, equals the result of subtracting four from seven times the number." Express this statement in algebraic notation and find the number to which it refers. (N.C.T.E.C.)

39. Find the value of x when

$$1 + 0·0042x = \frac{57·5 \times 1·063}{52·5}.$$ (U.L.C.I.)

Section B

Simultaneous Equations

Solve the following equations for x and y and verify the results.

1. $2x + 3y = 5.$
   $x + y = 2.$
2. $3x - 2y = 7.$
   $x + 2y = 5.$
3. $x + 4y = 6.$
   $2x - 3y = -15\frac{1}{2}.$
4. $8x - 3y = 39.$
   $7x + 5y = -4.$
5. $\frac{x}{3} + \frac{4}{y} = 6.$
   $\frac{x}{6} - \frac{y}{8} = 0.$
6. $\frac{x}{12} - y = -1;\ x - 6y = 0.$
7. $0·1x + 0·2y = -0·2.$
   $1·5x - 0·4y = 10·6.$

Section C

Miscellaneous Problems and Equations

1. Solve for P and Q

$$2P - 5Q = 2$$
$$3P + 10Q = 8{\cdot}6$$

2. Solve for P and Q

$$\frac{1}{P} + \frac{1}{Q} = \frac{7}{12}$$
$$\frac{1}{P} - \frac{1}{Q} = \frac{1}{12}$$

3. Find the values of $\frac{1}{x}$ and $\frac{1}{y}$ which satisfy the equations

$$\frac{2}{x} - \frac{3}{y} = 10$$
$$\frac{-3}{x} + \frac{5}{y} = 9$$

4. In a technical college 150 students were attending evening classes. Some attended 2 evenings a week for 3 hours an evening and the others 3 evenings a week for $2\frac{1}{2}$ hours an evening. If in a week the total number of hours attended by the 150 students was $1042\frac{1}{2}$, how many attended 2 evenings and how many 3 evenings per week? (Rugby.)

5. The wages of a plumber and an apprentice are in the ratio 2 : 1. Their weekly expenditures are in the ratio 13 : 6. If each saves 11*s*. a week, find their weekly wages. (Rugby.)

6. (*a*) Solve $6(x - 1) - 5(x - 2) = 4(x - 3)$.

(*b*) Simplify $\frac{x^2 - y^2}{x^2y} \div \frac{xy + y^2}{x^3y^2}$.

(*c*) In an isosceles triangle the base is two-thirds of one of the equal sides, and the sum of the sides is 40 in. Find the lengths of the sides. (Rugby.)

7. (i) Solve the equation

$$x^2 + 3x + \tfrac{1}{2} = 0.$$

(ii) Solve the simultaneous equations

$$\tfrac{1}{2}x + \tfrac{1}{3}y = 3$$
$$\tfrac{7}{8}x - \tfrac{1}{2}y = 2$$

(Sunderland.)

8. (*a*) Solve the equation

$$2x - 15y = 3x - 24y = 1.$$

(*b*) Given $R = A + \frac{v^2}{B}$, and that when $R = 8$, $v = 30$; and when $R = 12$, $v = 40$, evaluate the constants A and B. (U.L.C.I.)

9. It takes a car 15 min to overtake a car 4 miles in front of it. If the cars were coming towards each other they would meet in 4 min. What are the speeds of the two cars? (Coventry.)

10. A rectangular brass plate 10 in. $\times$ 12 in. is to have six bolt holes drilled in it of equal diameter. Calculate the largest hole diameter possible if the area of material left must not be less than half the area drilled away. (Nuneaton.)

11. Solve the equation $2x - 15y = 3x - 24y = 1$.

12. When an effort E lb is applied to a certain machine, it is found that a resistance R lb can be overcome, and that E and R are connected by the formula $E = a + bR$.

An effort of 3·5 lb overcomes a resistance of 5 lb, while an effort of 5·3 lb overcomes a resistance of 8 lb.

Find a and b and the effort required to overcome a resistance of 10 lb.

13. $y = ax^2 + bx^3$. When $x = 2$, $y = 5{\cdot}6$ and when $x = 3$, $y = 25$. Find the values of a and b.

14. Find two numbers such that the first is $2\frac{1}{2}$ times as great as the second, and the sum of both numbers exceeds half the first by 36. (U.E.I.)

15. Find the value of P and $\frac{Q}{P}$ from the following equations:

$$P + 3\left(\frac{Q}{P}\right) = 10$$
$$2P - \left(\frac{Q}{P}\right) = 6$$

Hence find the value of $\frac{P(Q-2)}{Q}$. (U.E.I.)

16. Find the values of $\frac{1}{x}$ and $\frac{1}{y}$, given that

$$\frac{4}{x} - \frac{1}{y} = 13$$

and

$$\frac{3}{x} - \frac{2}{y} = 6$$

Then find the values of x and y and of $\frac{y-x}{y+x}$.

(N.C.T.E.C.)

17. Solve the following equations:

$$x + \frac{4}{y} = 11$$
$$2x - \frac{1}{y} = 4 \qquad \text{(U.L.C.I.)}$$

CHAPTER 6

HARDER FORMULÆ—CONSTRUCTION—EVALUATION AND TRANSFORMATION

At the discretion of the teacher students may proceed directly to the Miscellaneous Exercises on Formulæ which commence *on p. 126.*

1. Construction of Formulæ

In Chapter 2, we dealt with the construction of simple formulæ. We now proceed to more difficult examples.

Example 1. *If* 1 *in.* = 2·54 *cm, and* 1 *lb* = 453·6 *gm, express* M *kg per litre in lb per cu ft.*

It is required to find the weight of 1 cu ft in lb and in terms of M.

(1) $1 \text{ kg} = 1000 \text{ gm} = \frac{1000}{453 \cdot 6}$ lb.

(2) Since 1 in. = 2·54 cm, 1 cu in. = $2{\cdot}54^3$ cc

and therefore 1 *cu ft*

$$= 2{\cdot}54^3 \times 1728 \text{ cc}$$
$$= \frac{25{\cdot}4^3 \times 1728}{1000} \text{ litres}$$
$$= 2{\cdot}54^3 \times 1{\cdot}728 \text{ litres}$$

Hence the problem resolves itself into finding the weight of $2{\cdot}54^3 \times 1{\cdot}728$ litres in lb.

Since 1 litre weighs M kg

1 cu ft weighs $M \times 2{\cdot}54^3 \times 1{\cdot}728$ kg

that is, 1 cu ft weighs $M \times 2{\cdot}54^3 \times 1{\cdot}728 \times \frac{1000}{453{\cdot}6}$ lb.

Simplified, this result becomes:

$$\frac{2{\cdot}54^3 \times 1728}{453{\cdot}6} \text{ M lb}$$

that is, 62·4 M lb.

$\therefore$ **M kg per litre = 62·4 M lb per cu ft.**

Example 2. *Express a velocity of* V *m.p.h. in feet per second*

$$\begin{aligned} \text{V m.p.h.} &= 1760 \times 3 \text{ V ft per hr} \\ &= \frac{1760 \times 3}{60 \times 60} \text{ V ft per sec} \\ &= \tfrac{22}{15} \text{ V ft per sec.} \end{aligned}$$

Example 3. *A rectangle has sides a in. and b in. long. The side of length a in. is increased by t in. and that of length b in. is diminished by t in. Establish an expression:*

(1) *For the change in area.*

(2) *For the approximate change in area if t is very small compared with a and b.* (U.E.I.)

(1) Original area $= ab$ sq in.

New area $= (a + t)(b - t)$ sq in.

$= (ab + bt - at - t^2)$ sq in.

Let $C =$ the change in area.

Then $C = ab - (ab + bt - at - t^2)$

that is, $C = at - bt + t^2$

(2) If t be small, t^2 will be very small and can be neglected.

Then $C = at - bt$ (approx.)

or $C = t(a - b)$

Example 4. *The breadth and height of a rectangular block are equal. Its length is five times its breadth. Obtain a formula for its total surface area in terms of its height.* (N.C.T.E.C.)

Let the height be h units of length.

Then the breadth $= h$, and the length $= 5h$ units of length.

The surface consists of four rectangles, the length of each being $5h$ units and breadth h units, together with two squares each of whose sides is h units.

Let S corresponding area units = the total surface area:

Then $S = 4(5h \times h) + 2h^2$

that is, $S = 22h^2$.

2. Evaluation of Formulæ

When the student requires to find the value of a formula corresponding to given values of the letters contained in it, he should carefully examine the formula in order to ascertain whether it is possible to change it to a form more suitable for calculation.

This he can often do by employing some of those algebraical operations which he has studied.

For example, if it were required to evaluate

$$A = \pi R^2 - \pi r^2$$

where $\pi = 3{\cdot}142$, $R = 14{\cdot}65$ and $r = 12{\cdot}55$, direct substitution would involve tedious calculation.

By using the methods of factorisation shown in Chapter 4, the formula can be simplified as follows:

$$\begin{aligned} A &= \pi R^2 - \pi r^2 \\ \therefore\ A &= \pi(R^2 - r^2) \\ &= \pi(R + r)(R - r) \end{aligned}$$

This is now in a much easier form for substitution. Similar devices will be indicated in the worked examples which follow.

Example 1. *The Simple Interest formula is expressed in the form—*

$$A = P\left(1 + \frac{rn}{100}\right)$$

where A *represents the amount in pounds at the end of n years. Find* A *if* P = 135, r = 4·5, *and* n = 4.

Substituting for the values given—

$$A = 135(1 + \tfrac{18}{100})$$
$$= 135 \times 1{\cdot}18$$

Thus, the amount $= £159{\cdot}3$
$$= £159 \text{ 6s.}$$

Example 2. *Given that* $\pi = 3{\cdot}142$, $D = 28{\cdot}6$ *and* $d = 11{\cdot}4$, *find* A *if* $A = \dfrac{\pi D^2 - \pi d^2}{144}$.

First factorise.

Then
$$A = \frac{\pi(D^2 - d^2)}{144}$$
$$= \frac{\pi(D + d)(D - d)}{144}$$

On substituting, $A = \dfrac{3{\cdot}142 \times 40 \times 17{\cdot}2}{144}$

Then if d and D are lengths measured in feet,

$$A = 15{\cdot}01 \text{ sq ft.}$$

Example 3. *Given that* $M = \dfrac{Cl^2 - l}{9\sqrt{3}}$ *find the value of* M *if* C = 15 *and* l = 1·4.

Factorising and multiplying both numerator and denominator by $\sqrt{3}$ in order to simplify the working (see Chapter 1, p. 23), we have:

$$M = \frac{l(Cl - 1)\sqrt{3}}{27}$$

Substituting, $M = \dfrac{1{\cdot}4(15 \times 1{\cdot}4 - 1)1{\cdot}732}{27}$

$$= \frac{1{\cdot}4 \times 20 \times 1{\cdot}732}{27}$$
$$= 1{\cdot}8 \textit{ approx.}$$

3. Changing the Subject of a Formula

Now that we have dealt with equations containing one or two unknowns, and have established the rules governing their solution, we can proceed to the manipulation of formulæ which involve several quantities.

Usually in a formula **one** of the symbols is expressed in terms of other symbols and each of the symbols has its own special use and meaning in the formula.

The **single symbol** thus expressed is called the **subject** of the formula.

Sometimes it is found necessary and convenient to make one of the other symbols the subject.

The student has already seen simple examples of this in Chapter 2, but the following examples, which have been worked out in detail, deal with transformations of greater difficulty.

Example 1. *In a cylinder in which* $h \equiv$ *the height,* $r \equiv$ *radius of the base and* $S \equiv$ *the total surface area, it is known that*

$$S = 2\pi r^2 + 2\pi rh$$

Express h *in terms of the other quantities.*

Take the term containing h to the L.H. side and the symbol S to the R.H. side.

Then
$$-2\pi rh = 2\pi r^2 - S$$

Change signs throughout.

$$2\pi rh = S - 2\pi r^2$$

Divide by the coefficient of h.

Then $$h = \frac{S - 2\pi r^2}{2\pi r}$$

Example 2. *If* $\dfrac{1}{R-1} = \dfrac{3}{1-t} - \dfrac{4}{1+t}$, *obtain the formula for* R, *and find its value when* $t = 0{\cdot}6$.

(U.L.C.I.)

In this case, first simplify the R.H. side, making *one fraction* of it.

Then $$\frac{1}{R-1} = \frac{3(1+t) - 4(1-t)}{1-t^2}$$

that is, $$\frac{1}{R-1} = \frac{3 + 3t - 4 + 4t}{1-t^2}$$

or $$\frac{1}{R-1} = \frac{7t-1}{1-t^2}$$

Inverting both sides,

$$R - 1 = \frac{1-t^2}{7t-1}$$

that is, $$R = \frac{1-t^2}{7t-1} + 1$$

$$= \frac{1 - t^2 + 7t - 1}{7t-1}$$

$$= \frac{7t - t^2}{7t-1} \quad \text{or} \quad \frac{t(7-t)}{7t-1}$$

Making the substitution $t = 0{\cdot}6$,

$$R = \frac{0{\cdot}6(7 - 0{\cdot}6)}{4{\cdot}2 - 1}$$

$$= \frac{0{\cdot}6 \times 6{\cdot}4}{3{\cdot}2}$$

$$= 1{\cdot}2$$

This example could have been worked, as a first step, by multiplying throughout by the common denominator $(R-1)(1-t)(1+t)$, but this method involves more operations.

Example 3. *Given that* $E = \dfrac{0{\cdot}0007lv^2}{d}$, *express* $\dfrac{d}{l}$ *in terms of* E *and* v. *State the effect on the value of* $\dfrac{d}{l}$ *of:*

(1) *halving the value of* E;
(2) *doubling the value of* E.

(N.C.T.E.C.)

Divide both sides by $0{\cdot}0007v^2$.

Then $$\frac{E}{0{\cdot}0007v^2} = \frac{l}{d}$$

Inverting both sides and changing over,

$$\frac{d}{l} = \frac{0{\cdot}0007v^2}{E}$$

The denominator of the R.H. side consists of E only. Hence if E be halved, the whole fraction, and therefore $\dfrac{d}{l}$, will be doubled. If E be doubled, $\dfrac{d}{l}$ will be halved.

EXERCISE VI

1. Construction of Formulæ

1. A rectangular piece of metal is a in. by b in. Its weight is c lb. Write down expressions for:

(a) The weight in oz per sq in.
(b) The weight in lb per sq ft.

2. The perimeter of a square is $4x + 16$ in. Write down an expression for:

(a) The length of one side of the square in in.
(b) The area of the square in sq in. (U.L.C.I.)

3. Express a velocity of K cm per sec in miles per hour (take 1 km = 5 furlongs).

4. Express x in.-tons per sec in ft-lb per min. (N.C.T.E.C.)

5. Express

(1) x knots in terms of ft per sec, given that 1 nautical mile = 6,080 ft, and 1 knot = 1 nautical mile per hr.

(2) p lb per sq in. in terms of gm per sq cm, given that 1 in. = 2·54 cm, 1 oz. = 28·35 gm. (N.C.T.E.C.)

2. Evaluation of Formulæ

1. Assuming that $F = 4\pi I\left(1 - \frac{h}{a}\right)$, find F if $\pi = 3{\cdot}142$, $I = 7300$, $h = 3{\cdot}6$ and $a = 8{\cdot}4$.

2. The current C in a certain conductor is given by the expression

$$C = \frac{0{\cdot}108}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}}$$

Calculate C when $r_1 = 8$, $r_2 = 10$, $r_3 = 12$ and $r_4 = 14$. (U.E.I.)

3. If $W = 0{\cdot}532a(D^3 - d^3)$, find its value when $d = 6$, $a = 10$ and $D^3 = 279{\cdot}65$.

4. The relation between the temperature on a Fahrenheit thermometer and that on a Centigrade thermometer is expressed by the formula $F = \frac{9}{5}C + 32$.

Express a temperature of 27·5° C. in Fahrenheit degrees.

5. If $C = \frac{E}{R + r}$, find C when $E = 16{\cdot}5$, $R = 2{\cdot}8$, $r = 1{\cdot}3$.

6. If $C = \frac{E + e}{R + r}$, find C if E = 17·6 volts, e = 1·5 volts, R = 28·4 ohms and r = 2·6 ohms.

7. If p is the pressure in a thin pipe of outside diameter d and thickness t, the greatest tensile stress being f, then

$$t = \frac{pd}{p + 2f}$$

Find t when $f = 4{,}000$, $p = 500$ and $d = 8$.

9. If $E = \frac{Wv^2}{2g}$ find E when $W = 15{\cdot}5$, $v = 18{\cdot}8$ and $g = 32$.

8. Being given that $f = \sqrt{\frac{p^2}{4} + q^2}$, find the value of f when $p = 12$ and $q = 8$. (U.L.C.I.)

3. Changing the Subject of a Formula

The student should note with care that formulæ such as are dealt with in this chapter are valid only if the quantities concerned are measured in the appropriate corresponding units.

For a formula such as that for the volume of a rectangular solid

$$V = l \times b \times t$$

it is only necessary that l, b and t should each be numbers of the same length unit, when it can be taken for granted that V is a number of *corresponding* volume units. For such a formula, however, as the horse-power formula given below—

$$H = \frac{EC}{825}$$

the number 825 arises from the fact that E and C are measured in particular units, and these units must be employed in any application of the formula. In fact, the use of this formula implies that E and C are numbers of volts and amperes respectively, while the constant 825 also embodies some assumed value for the efficiency of the motor.

1. If $C = \frac{E - e}{R}$, find E in terms of the other quantities and calculate E when $C = 100$, $e = 240$, $R = 0{\cdot}05$.

2. In the formula $T = \frac{\pi f d^3}{16}$ find

(a) f in terms of the other quantities,
(b) d in terms of the other quantities.

3. The horse-power of a motor is given by the formula

$$H = \frac{EC}{825}$$

Express this as a formula for C in terms of the other letters.

4. If $H = 0{\cdot}5d^2(r + 1)$, express this as a formula for (1) d, (2) r in terms of the other quantities.

5. If $a + \frac{b}{q - np} = C$, express this as a formula for n in terms of the other quantities.

6. Given that $r = \frac{R(E - V)}{V}$, express V in terms of r, R and E.

State the effect on the value of V of doubling the value of E. (N.C.T.E.C.)

7. The velocity V of water in a pipe occurs in the following formula:

$$h = 0{\cdot}03\,\frac{L}{D} \times \frac{V^2}{2g}$$

Change round the expression so as to make it more suitable for the calculation of V. Then calculate V when $h = 0{\cdot}614$, $L = 168$, $D = \frac{1}{4}$ and $g = 32{\cdot}2$.

Without working out, state the effect on V of doubling the value of L. (U.E.I.)

8. Given $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$, find R′

when $R_1 = 8{\cdot}6$, $R_2 = 4{\cdot}3$, $R_3 = 2$. (U.L.C.I.)

9. Given $28t(p - d) = \frac{23\pi d^2}{4}$, find p when $t = 0{\cdot}5$, $d = 1{\cdot}2\sqrt{t}$. (U.L.C.I.)

10. Given $n^2r + 1 = NR$, rearrange the terms so as to find the value of n.

Calculate n when $r = 0{\cdot}725$, $N = 14$, $R = 2{\cdot}73$.

11. The stress f in the material of a thick cylinder is given by the formula

$$\frac{D}{d} = \sqrt{\frac{f + p}{f - p}}$$

(a) Express f in terms of the other quantities.
(b) Calculate f when $p = 1500$ lb per sq in., $d = 9{\cdot}75$ in., $D = 19{\cdot}75$ in., and state the units in which f is expressed. (U.E.I.)

12. The lifting force of an electro-magnet is given by the formula

$$F = \frac{B^2A}{112 \times 10^5}$$

where F is the force in lb, A is the area of the pole face in sq cm, and B is the flux density in lines per sq cm.

(1) Change the formula round to express B in terms of the other quantities.
(2) Find the value of A when $F = 85{\cdot}6$ lb and $B = 10{,}500$. (U.E.I.)

13. Given that $v^2 = u^2 + 2fs$ express f in terms of v, u and s. (N.C.T.E.C.)

14. Given $a(P - \frac{1}{2}Q) = b(Q - \frac{1}{2}P)$ rearrange the terms so as to express P in terms of the other quantities. From the rearranged equation calculate the value of P when $a = 3$, $b = 1{\cdot}5$ and $Q = 270{\cdot}5$. (U.L.C.I.)

15. (a) Given $S = \sqrt{\frac{3L(L - x)}{8}}$, find the value of x when $S = 2$ and $L = 45$.

(b) Given $I = \frac{nE}{R + nr}$, find the value of n when $I = 2$, $E = 1{\cdot}8$, $R = 2{\cdot}4$, $r = 0{\cdot}5$. (U.L.C.I.)

16. Given that

$f = \frac{2(s - ut)}{t^2}$, express u in terms of f, s and t.

Find the value of u when $s = 80, f = 32, t = 2{\cdot}5$.

(N.C.T.E.C.)

17. The amount of sag d in a beam under certain loading is given by the expression

$$d = \frac{Wl^3}{48EI}$$

Change round the formula so as to express l in terms of the other quantities. (U.E.I.)

18. The diameter (D in.) of a shaft subjected to twisting stress occurs in the following formula:

$$A = \frac{583TL}{ND^4}$$

Change round the formula so as to express D in terms of the other quantities. (U.E.I.)

Miscellaneous Exercises

Mainly from examination papers set in connection with National Certificate courses. No. 1 is, however, based upon a C.G.L.I. Intermediate examination question.

1. Washers (metal discs pierced with a central hole) are often made from sheet metal in a punching press. A large number of washers is required whose dimensions are to conform to the formula $D = 1\frac{1}{2}d + \frac{1}{8}$, where D and d are the outside diameter and the diameter of the hole respectively, both being measured in inches, since otherwise the term $\frac{1}{8}$ would be meaningless. The makers know from experience that to secure an accurate punching a clearance of $\frac{1}{4}$ in. must be left between any two discs, and between any disc and the edge of the sheet or strip. For each of

the following cases, Figs. (i) to (iv), make a good-sized clear sketch showing with all necessary dimensions the lay-out of the washers, and build up a formula giving as a percentage the ratio:

$$\frac{\text{Weight (or area) of metal sold as washers}}{\text{Weight of metal in original strip or sheet}}$$

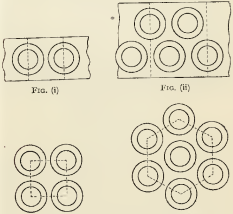


Fig. (i) Fig. (ii)

Fig. (iii) Fig. (iv)

Fig. 22.

Work out the actual percentage for washers having $1\frac{1}{16}$ in. dia hole (to give a clearance when slipped over 1 in. dia bolts).

Particulars of Figs. (i) to (iv)

Fig. (i). The material is steel strip just wide enough to give the necessary clearance on each side of a single disc,

and so long that the interruption of the pattern at the ends need not be taken into account.

Fig. (ii). The material is steel strip wide enough to contain two discs staggered as in the figure. Again the ends need not be taken into account.

Fig. (iii) and Fig. (iv). The material is in quite large sheets, so large that the interrupted pattern at the edges need not be taken into account. (C.G.L.I.)

2. Devise a formula for the length l ft of $\frac{3}{4}$ in. dia bar which can be rolled from a billet a in. long and b in. square.

3. An angle section $\frac{3}{4}$ in. $\times$ $\frac{3}{4}$ in. $\times$ $\frac{1}{8}$ in. is "extruded" from an aluminium alloy billet d in. dia by l in. long. What is the maximum length that can be obtained?

4. Make x the subject of the following equations:

(a) $y = 2\pi\sqrt{\dfrac{L^2 + x^2}{gL}}$.

(b) $y = \dfrac{4\pi}{3}(a^3 - x^3)$.

(c) $y = (x^{5/2} + 1)$. (Rugby.)

5. The formula for the time of swing of a simple pendulum is $T = 2\pi\sqrt{\dfrac{L}{g}}$. Find L when $T = 10$, $\pi = 3{\cdot}142$, $g = 32{\cdot}2$. (Coventry.)

6. A cylinder and sphere have equal volumes. The radius of the sphere is equal to that of the cylinder. Find a formula for the height of the cylinder in terms of the radius R. (Coventry.)

7. If $s = \pi r\sqrt{r^2 + h^2}$, find h in terms of π, s and r. (Cannock.)

8. (i) If $h = r - \sqrt{r^2 - a^2}$, show that $a = \sqrt{h(2r - h)}$.

(ii) If $s = ut + \frac{1}{2}gt^2$, find g in terms of the other quantities, and its value when $s = 132$, $u = 12{\cdot}5$ and $t = 2{\cdot}5$. (Sunderland.)

9. The nominal horse-power of a motor car is given by $H = \dfrac{2nd^2}{5}$, where n is the number of cylinders and d the diameter of each cylinder in inches. Find the diameter of each cylinder of a four-cylinder engine of 11·9 h.p. If the diameter of each cylinder is increased by 10%, find the extra horse-power developed. (Sunderland.)

10. (a) If $E = d + \dfrac{Q^2}{2gd^2}$, develop a formula making Q the subject.

(b) If $R = R_0(1 + kt)$, find the value of R_0 when $R = 100$, $k = 0{\cdot}0043$ and $t = 50$. (U.L.C.I.)

(c) Rearrange the expression $p = 2t\left(1 - \dfrac{t}{r}\right)$ to give r in terms of p and t.

(d) The currents I and i in two arms of a circuit are connected by the following equations:

$$3I + 5i = 23$$
$$5I + 3i = 10$$

Find the values of I and i. (U.L.C.I.)

11. (a) If $t^2 = \dfrac{4\pi^2 ma}{(M + 2m)g}$, find the value of m in terms of M, g, a and t.

(b) (i) Express as a single term:

$$p^{-a} \times p^{2b} \times p^{-3c}.$$

(ii) Simplify giving answer with positive indices only:

$$(x^2 . \sqrt{y})^2 \times (\sqrt{x} . y^2)^{-2}$$

(Nuneaton.)

12. Rearrange the formula

$$i = \frac{nE}{R + nr}$$

so that (a) r is given in terms of i, R, n and E; (b) n is given in terms of i, R, r and E. (Nuneaton.)

13. (*a*) Eliminate v from the equations $v = u + ft$ and $v^2 = u^2 + 2fs$ and hence find an equation giving s in terms of u, f and t.

Find the value of s when $u = 40$, $t = 3{\cdot}5$ and $f = -32{\cdot}2$.

(*b*) The angle of a regular polygon of n sides is given by

$$\theta = 2\left(90 - \frac{180}{n}\right).$$

Make n the subject of this formula and find the number of sides of a regular polygon whose angle is 165°.

(E.M.E.U.)

14. Given that $\left(p + \frac{a}{v^2}\right)(v - b) = 1 + \frac{t}{273}$ calculate the value of the temperature t when $p = 42{\cdot}5$, $a = 0{\cdot}00874$, $b = 0{\cdot}0023$, $v = 0{\cdot}01$. (Dudley.)

CHAPTER 7

GRAPHICAL WORK

1. In presenting and comparing quantities of the same kind, advertisers and statisticians frequently resort to pictorial illustrations, which, at a glance, afford the public an easy means of understanding and appreciating the deductions to be drawn from those quantities.

For example, the populations of various countries may be compared by means of areas of squares. Exports in various years may be shown by rectangles of equal base, but of varying height. Vertical lines may be used to make a comparison of varying temperatures, and so on.

Example. Practically everyone is familiar with what we call a **temperature chart.** See Fig. 23.

The table appended below gives the temperature of the air at 12.0 *noon on six successive days.*

Day.	May 1.	2.	3.	4.	5.	6.
Temperature (° F.) .	58	65	62	64	70	55

The two lines OX and OY are called **the axes of reference.** On OX take points at equal distances to indicate the days and draw the vertical lines to illustrate the temperatures corresponding to those days. In such a chart it will be noticed that the points which mark the temperature levels are joined by a series of short lines.

The result is a more or less wavy line, which, apart from showing the actual temperature, does convey pictorially some idea of the rises and falls in temperature.

Such a **graph,** as we may call this wavy line, together with others dealing with statistics or experimental values, presents a pictorial form of comparing magnitudes of the same kind.

In **graphical work,** as we know it in Mathematics, when we wish to represent a set of statistics, we dispense with the

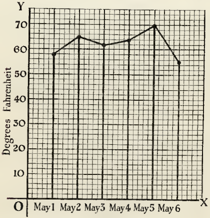


Fig. 23.

idea of drawing vertical lines, since they appear on the squared paper, and content ourselves with marking the **positions** on these lines.

It must be noted, however, that a complete study of **graphs** implies more than a pictorial representation, and this it will be our purpose to show later on.

2. Graphs Relating to Statistics and Experimental Data

We shall now take examples of statistical and experimental graphs, in an endeavour to discover whether the graph suggests any law connecting the values plotted.

Example 1. *The following table gives the values of exported manufactured goods of a certain type in certain specified years.*

Year.	1950.	1951.	1952.	1953.	1954.	1955.
Value in millions .	6·0	5·2	4·9	5·1	4·8	5·6

Show the variations in the values of the goods by a graph.

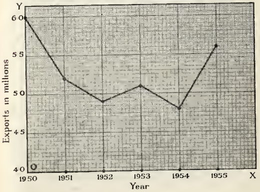


Fig. 24.

We first draw two **axes of reference,** OX and OY, at right angles, and mark off along the horizontal axis OX equal distances to indicate the successive years as shown above (Fig. 24).

In order to make the differences in value as pronounced as possible, we choose a fairly open scale for the vertical axis by starting with 4 millions at O, as the values we have to deal with lie between 4 and 6 millions.

Corresponding to the years, mark the points which give the values of the goods according to the vertical scale and connect these points by a line drawn as evenly as possible.

An examination of this graph shows a rather sudden drop in values from 1950 to 1951. From 1951 to 1952 the fall is continued but not so rapidly, followed by a slight rise, and then a slight fall in 1953 and 1954.

From 1954 to 1955 there is a somewhat abrupt rise.

Evidently the rises and falls do not follow any set plan, or obey any definite rule.

Example 2. *The average weight of boys of different ages is given in the following table. Draw a graph to illustrate.*

Age in years	11	12	13	14	15
Weight in lb	80	85	92	101	114

As in the previous case, draw two axes at right angles, indicating the age on the horizontal axis (Fig. 25).

Since the weights range between 80 lb and 115 lb we can make 75 lb the starting value at O for the vertical axis.

This curve does not present the irregularities we have in Example 1. There is a general tendency for the curve or graph to rise, year by year, from the lowest value to the highest, so that in some way or other **the weight depends on the age.**

This is clearly a case which exhibits a certain degree of regularity, and, that being so, we can employ the curve to deduce weights corresponding to intermediate ages.

For example, if we take the age of $12\frac{1}{2}$ years, find the point A on the graph which corresponds and then the point

C on the vertical scale corresponding to A, we find that a boy $12\frac{1}{2}$ years old will on the average weigh 88·3 lb.

Using a curve in this way to find values which have not been given, but which are derivable from the curve itself, is called **Interpolation.**

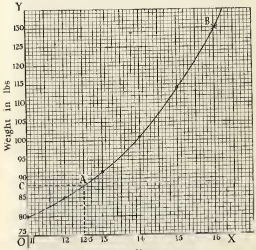


FIG. 25.

It must be noted that it is the regular tendency of the curve which points to the probability of these intermediate values being more or less correct.

Also if we extend the curve and follow its general trend, we may find the point which probably indicates the average

weight at an age of 16 years not contained in the statistics provided.

The point B thus found indicates a probable average weight of 130 lb at the age of 16 years.

Finding a probable value in this way is termed **Extrapolation.**

Example 3. Now let us take an example from Experimental Data.

Below are given the weights of potassium bromide which will dissolve in a given volume of water at a certain temperature. Draw a graph to illustrate.

Temperature ° Cent.	0°	20°	40°	60°
Weight of bromide in gm	2·67	3·23	3·73	4·24

Proceeding as in the previous cases, we obtain the graph as shown (Fig. 26).

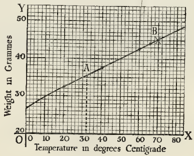


Fig. 26.

This graph is characterised by a steady rise from left to right, and forms approximately one straight line.

Such irregularity as exists is very slight, and may be ascribed to experimental errors. Evidently there is some law connecting the amount dissolved with the temperature; in other words, the **amount dissolved** depends in some definite way on the **temperature.**

By **Interpolation** from the graph we find that at 32° approximately (see point A), 3·55 gm will dissolve.

Extending the graph as in the previous case, we also find that at 70° (B) the amount which will probably be dissolved is 4·5 gm.

Example 4. *In the table below are given lengths of wire of the same material and cross-section with the corresponding resistances in ohms.*

Lengths in yd	100	120	170	220
Resistance in ohms	2·5	3	4·25	5·5

Draw the graph and find the resistance for a length of 155 *yd.*

In this case we will take our starting-point on the **horizontal** axis at 100 yd, and proceed as in the previous cases (Fig. 27).

The points plotted from the data are seen to lie on a **straight line** which runs uniformly from left to right.

Evidently the **number of ohms depends** on the **length of the wire,** and the two variable quantities must be connected by a definite law.

Example 5. *The following table gives the distances which a body travels from rest with the corresponding times.*

Draw a graph to show the relation between the times and the distances.

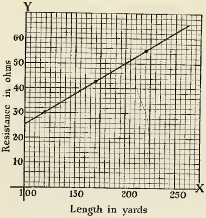


FIG. 27.

Times in sec .	0	1	2	3	4	5
Distance in ft	0	2	8	18	32	50

In this example, since the distances cover a fairly wide range, the unit of distance on the vertical scale must be small in comparison with the time unit on the horizontal axis.

The values thus plotted do not give a straight line (Fig. 28) but a curve, which has a definite form, shows considerable regularity and apparently indicates that there is some definite connection between the **time** and the **distance.**

In other words, there must be a law connecting the two quantities.

Example 6. *The following table gives certain temperatures in degrees Centigrade (C.) with the corresponding Fahrenheit values (F.).*

Illustrate by a graph.

C. . . .	−30	−20	0	10	20
F. . . .	−22	−4	32	50	68

It will be noticed that this case is unlike the others, since some of the values both for C. and F. are negative.

This will necessitate the **axes of reference** being extended to the left and downwards as shown on p. 140 (Fig. 29).

Along OX and OY mark off units in a positive sense, as in the previous cases.

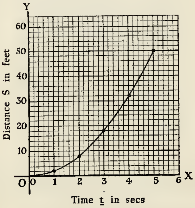


FIG. 28

Along OX_1 and OY_1 mark off in a negative sense for Centigrade and Fahrenheit respectively.

On joining the points whose distances from the axes YOY_1 and XOX_1 give the corresponding temperatures

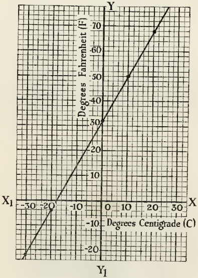


FIG. 29.

tabulated above, it is found that they lie on a straight line, indicating that the temperatures are connected by a definite law.

This we know to be the case, and the relation between F. and C. can be expressed by $F. = \frac{9}{5}C. + 32$.

3. Sufficient examples have now been given to show that the plotting of statistics, or observations, may produce graphs which can be divided into three groups:

(1) Those which possess no regularity, and which do not follow any definite law.

(2) Those which appear to show some connection between the two sets of values, and in which there seems to be **dependence** of one set of values on the other set.

(3) Those which give a straight line, or a definite regular curve, and so point to the existence of a definite law.

When such a law exists, one set of values depends entirely on the other set, value for value.

We may now deduce that every straight line or regular curve obtained by plotting values of two variables one against the other is evidence of a definite law connecting the two variables.

We see also that if we are given a law which connects two variables, we can draw the straight line or regular curve which corresponds.

4. From the graphs so far considered, in which a law of some kind connects the two quantities plotted, we may in general conclude that one of these quantities depends for its value upon the other.

Thus in Example 5, the distance travelled depends upon the time; in Example 4 the resistance depends upon the length of the wire. Of the two variable quantities the one which is thus dependent upon the other is called the **dependent variable,** and the other is called the **independent variable.**

When we generalise it is usual to denote the **independent variable by x,** and **the dependent by y.**

On the actual graphs it is customary to mark values of the independent variable x along a line such as OX (Fig. 28),

which is called the x axis, and values of the dependent variable y upon OY at right angles to OX and called the y axis. The intersection of these axes (O) is called the **origin.**

We will now consider some examples of graphs which are straight lines and investigate the law which connects the two variables in these cases.

Case I

In Fig. 30, AOB is a straight line graph which bisects the angle between the axes OX and OY.

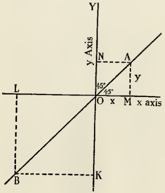


FIG. 30.

Let A be any point on the straight line.
Draw AM perpendicular to the x axis.

Then $\angle AOM = 45°$
Also $\angle OAM = 45°$
$\therefore\ AM = OM$

Let y denote the distance of A from the x axis,
and let x ,, ,, ,, ,, A ,, ,, y axis.

Then for the point A, $y = x$.

This is clearly also true for B and for any other point on the line.

Consequently AOB is such a straight line that for any point on it, the law connecting the distances of the point from the axes can be expressed by

$$y = x$$

This is called the Equation of the Line.

Case II

In Fig. 31 the line MN passes through the origin O so that every point on it has its value for y double that of the corresponding value for x.

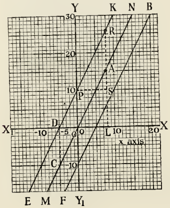


FIG. 31.

For example, AL = 2OL for the point A,
and CD = 2OD for the point C.

Hence we say that the law connecting x and y for the line MN is $y = 2x$.

Case III

If now we have a line EK parallel to that in Case II (Fig. 31) but which has corresponding points moved up 10 units, we see that the y value is 10 units greater for the same x value. In the same figure, A has moved up to the point R.

Thus RL = RS + SL
and RS = 2SP = 2OL = $2x$
and SL = 10
∴ For this line y = RL = $2x + 10$

Similarly the line FB parallel to these, and passing through the point -10 on the line OY_1 would be expressed by the equation

$$y = 2x - 10$$

Case IV

In Fig. 32 the line MN bisects the $\angle X_1OY$ and slopes upwards to the left. For the point A on it, the x value $= -10$ and the y value $= +10$.

Similarly for any other point.

Hence the law for the line MN is $y = -x$.

Then, as in the previous case, the equation for the line SR which is parallel to MN and with corresponding points moved up 10 units is

$$y = -x + 10$$

Similarly the line KL, which is parallel to MN and SR and which passes through the point -5 on the axis OY_1, would be expressed by the equation

$$y = -x - 5$$

5. These examples could be multiplied indefinitely. Finally by generalising we see that **all straight lines** drawn with reference to axes in this way show a relation between two variables which are connected by a law of the form

$$y = mx + b.$$

The numbers m and b are constants, and will depend upon the particular line we have under consideration.

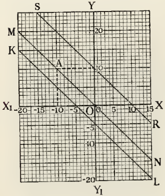


FIG. 32.

It will be seen that b is given by the distance on the y axis between the origin and the point where the straight line cuts that axis; or, more briefly, b is called the **intercept on the y axis.**

The meaning of m will be apparent later (see Chapter 10).

NOTE.—If $b = 0$, the straight line passes through the origin and the equation becomes $y = mx$.

The equation $y = mx + b$ is of the first degree in x and y. It will be readily seen that the converse of the above is true. Thus if any equation connecting two variables x and y is of the first degree in these, the graph obtained by plotting corresponding values of x and y will always be a straight line. Hence the law connecting two quantities, one of which is dependent on the other, and the graphical expression of which is a straight line, is called a **Linear Law.**

6. To draw a Straight-line Graph when the Law is Given

1. *Draw the graph of* $y = 1{\cdot}5x - 3$.

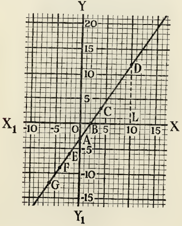


Fig. 33.

Draw axes XOX_1 and YOY_1 at right angles as shown (Fig. 33). From the origin O set off units to any desired

scale. The units need not be necessarily the same on each axis.

Since the equation $y = 1{\cdot}5x - 3$ is a definite statement of the relation existing between x and y, for any assumed value of x we can find the corresponding value of y by substitution.

This has been done, and the result of the substitutions is set out below.

	A	B	C	D	E	F	G
When $x =$	0	2	4	10	-2	-4	-6
$y =$	-3	0	3	12	-6	-9	-12

Each pair of values of x and y gives one point on the line, and in this case, in order to assist the explanation, each point has been denoted by a letter A, B, C, etc.

In plotting these points it must be remembered that the x value is measured to the right or to the left of the y axis, and the y value is measured above or below the x axis.

Point A. Since $x = 0$, it must lie on the y axis, and since at the same time $y = -3$, it must lie 3 units below the x axis.

Point D. Since $x = 10$, it must lie 10 units to the right of the y axis, and since $y = 12$, it must lie 12 units above the x axis.

Point E. Since $x = -2$, it must lie 2 units to the left of the y axis, and since $y = -6$, it must lie 6 units below the x axis, and so on for the rest of the points.

We say that A is the point $(0, -3)$ and the values 0 and -3 are called its **co-ordinates.** They are placed within brackets as shown for every point thus indicated.

Similarly the **co-ordinates** of D are 10 and 12, and D is said to be the point (10, 12).

The line passing through all these points A, B, C, etc., thus determined will give the required graph.

It is very important to realise that the equation of the line **is satisfied by the co-ordinates of any point on the line**—that is, if they are substituted in the equation the two sides are equal.

2. *Show by a graph the relation between x and y in the equation*

$$2y + 5x = 8$$

Dividing each term by 2 and rearranging, we can put this equation into the standard form. This also renders it easier for substitution.

Then $\quad y + \frac{5}{2}x = 4$

that is, $\quad y = -\frac{5}{2}x + 4$

At this stage, it may be as well to point out that since we know that we are going to get a straight line, three points will be sufficient for our purpose. The third point acts as a check on the other two.

The table and the corresponding graph are shown in Fig. 34.

		B	A	C
When	$x =$	0	-4	6
	$y =$	4	14	-11

The points A, B and C are plotted as shown in the previous example.

The student should note that this graph runs downwards from left to right, and correspondingly the coefficient of x, namely $-\frac{5}{2}$, is a negative quantity.

Compare this with Example I, in which the graph runs upwards from left to right and the coefficient of x is positive, *i.e.* 1·5.

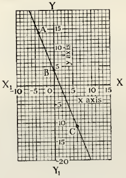


FIG. 34.

7. To Find a Graphical Solution to Two Simultaneous Equations of the First Degree

Example. *Solve graphically* (1) $2y + x = 5$
(2) $y - x = 1$

We have seen that to solve these equations we need to find a pair of values of x and y which will simultaneously satisfy both.

We have also seen that if we draw the graph of one of these lines, we can find any number of points whose co-ordinates will satisfy the equation, and so for the other line. If now we draw both lines on the same diagram, and find that they intersect, then the co-ordinates of this point

will satisfy **both** equations simultaneously, and give the solution which we require.

We can then proceed as follows:

Divide (1) throughout by 2 and rearrange both equations

Then (1) $y = -\frac{1}{2}x + 2\cdot5$

(2) $y = x + 1$

Adopting the plan explained in the previous section, we get the lines MN (Fig. 35), which is the graph of $y = x + 1$, and RS, which is the graph of $y = -\frac{1}{2}x + 2\cdot5$, or $2y + x = 5$.

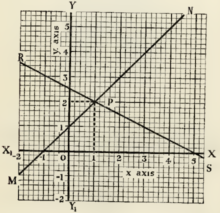


FIG. 35.

These two lines intersect at the point P, whose co-ordinates are seen to be (1, 2).

That is, at P $x = 1$, and $y = 2$.

Now, P lies on **both** straight lines and therefore its

co-ordinates satisfy the law for each of those lines—that is, $x = 1$ and $y = 2$ satisfy the equations $2y + x = 5$ and $y - x = 1$.

This can be confirmed if the equations are solved algebraically.

8. Given a Straight Line, to Determine its Equation—that is, the Law Connecting the Two Variables

Example.

Let MN (Fig. 36) be the given straight line whose equation in general form is $y = mx + b$.

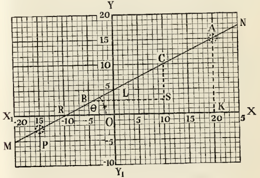


FIG. 36.

We have to determine the constants m and b.

Now, the co-ordinates of any point on this line satisfy the equation

$$y = mx + b$$

Take two points A and P not too close together and in positions where the co-ordinates can be easily determined.

(1) The co-ordinates of A are (20, 15).
(2) The co-ordinates of P are (— 15, — 2·5).

Using these values and substituting in $y = mx + b$, we have:

$$\text{(i)} \quad 15 = 20m + b$$
$$\text{(ii)} \quad -2{\cdot}5 = -15m + b$$

We now have two simultaneous equations with m and b as the unknown quantities.

Subtracting, $17{\cdot}5 = 35m$

$\therefore \quad m = \frac{1}{2}$

Substituting in (i),

$15 = 10 + b$

$\therefore \quad b = 5$

Hence the required **law** is

$$y = \tfrac{1}{2}x + 5$$

The following points should be noted in connection with the results just obtained:

(1) The graph cuts the y axis at L, the co-ordinates of which are (0, 5); the distance OL is the intercept on the y axis.

(2) Now, $\dfrac{OL}{OR} = \frac{5}{10} = \frac{1}{2}$.

Also $\dfrac{AK}{KR} = \frac{15}{30} = \frac{1}{2}$.

Further, take any point C in the graph and draw CS any distance parallel to the y axis. Then draw SB parallel to the x axis, meeting the graph again at B.

In this case $\dfrac{CS}{SB} = \frac{7}{14} = \frac{1}{2}$.

This result is the same wherever C be taken. The fraction $\frac{1}{2}$, which is a constant for the line, and which is seen to be equal to m, the coefficient of x, is called the **gradient** of the line MN.

The angle θ which the line MN makes with the axis of x measured in the positive direction is called its **angle of slope.** This angle can be correctly measured only when the units are the same on both axes.

In Fig. 33, for example, the intercept on the y axis is -3, and in the form in which the equation is written $b = -3$.

Again, the **coefficient** of x is 1·5 and $\dfrac{DL}{LB} = \frac{12}{8} = 1{\cdot}5$.

This, then, is the gradient of the line.

It follows from the above that the equation of a line can be determined when we know

(1) The intercept on the y axis,
(2) The gradient of the line.

We must, however, note that this alternative method is best adapted to the more simple cases, and those in which the values of the variables can be easily obtained from the given graph.

Example 2. *To find the equation of the straight line in Fig.* 37.

Let MN be the given straight line (Fig. 37).

In this example take two points, A and P, as before.

Their co-ordinates are (— 3, 3) and (1·5, — 3) respectively.

Then, substituting in the general equation $y = mx + b$, we have

$$(1) \quad 3 = -3m + b$$
$$(2) \quad -3 = 1{\cdot}5m + b$$

Subtracting, $6 = -4{\cdot}5m$

$$\therefore \quad m = \frac{-6}{4{\cdot}5} = -\tfrac{4}{3}$$

Substituting in (1),

$$3 = (-3 \times -\tfrac{4}{3}) + b$$
$$3 = 4 + b$$
$$\therefore \quad b = -1$$

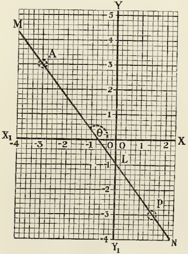


Fig. 37.

Hence the equation for the line is $y = -\frac{4}{3}x - 1$.

The intercept $OL = -1$, and the gradient is $-\frac{4}{3}$, a negative quantity.

It will be seen that in this case the line MN makes an angle θ with the positive direction of the axis of x where θ is greater than 90°.

9. Equation of a Line from Experimental Data

This method of determining the law connecting two variables as illustrated in the last two examples can be applied when we are furnished with data which are believed to be connected by a linear law.

Example 1. *In a series of experiments carried out with a Weston Differential Pulley Block the effort* E *lb necessary to raise a load* W *lb was found to be as follows:*

W	10	20	30	40	50
E	3·3	4·8	6·4	7·9	9·5

Show these values on a graph and determine the law which they seem to follow, and find the probable effort when the load is 25 lb.

Examining the data, and noting the maximum value to be shown in each case, we can take 0·5 in. on the horizontal axis to represent 10 lb for the load W, and 0·25 in. to represent 1 lb for the effort E.

Then plot the points as shown (Fig. 38). Since the data are derived from experimental results, slight deviations from a straight line are to be expected. If any one or two points are definitely not in accordance with the majority, the experiment should be repeated if possible in order to check them.

A straight line should be drawn to take in as many of the points as possible, or, failing that, it should be so drawn that the points are fairly evenly distributed on either side of it.

We now take on this line, two points A and B which are suitable for reading off the values. They will not necessarily be any of the points actually plotted, and it is advisable to choose them fairly wide apart.

The quantities E and W are evidently connected by a linear law which will be of the form

$$E = mW + b \qquad \text{[see § 5]}$$

For the point A, W = 35 lb and E = 7·2 lb.
For the point B, W = 12 lb and E = 3·6 lb.

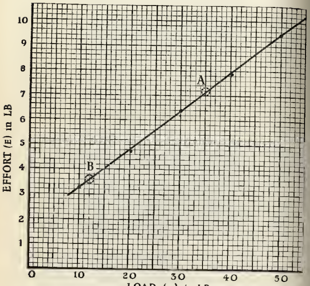


FIG. 38.

Hence, substituting in $E = mW + b$, because these values satisfy the required law, we have:

(1) $7{\cdot}2 = 35m + b$ (2) $3{\cdot}6 = 12m + b$

Subtracting, $3{\cdot}6 = 23m$

that is $m = \dfrac{3{\cdot}6}{23} = \dfrac{18}{115} = 0{\cdot}157$

$= 0{\cdot}16$ approx.

Substituting in (2),

$$3{\cdot}6 = 12m + b$$

$$3{\cdot}6 = \frac{12 \times 18}{115} + b$$

$$\therefore \quad b = 3{\cdot}6 - 1{\cdot}88$$

$$= 1{\cdot}72$$

Hence the law is $E = 0{\cdot}16W + 1{\cdot}72$.

To find E when the load is 25 lb, substitute in this law thus determined.

Then $E = 0{\cdot}157 \times 25 + 1{\cdot}72$

$= 5{\cdot}6$ lb approx.

This result agrees very closely with the graph itself.

Example 2. *When two voltmeters are compared, they have corresponding readings* C *and* K *as set out below.*

C . . .	1·9	2·75	3·8	4·8	5·8
K . . .	5·75	8·3	11·2	14	16·8

Find the relation between C *and* K.

In plotting these points the scale for C can be more open than that for K. Take 1 in. to represent 1 unit of C on the horizontal axis, and 1 in. on the vertical axis to represent 5 units of K. (The figure printed is reduced.)

The two quantities C and K are assumed to be connected by a straight-line law and this will be of the form

$$K = mC + b$$

The straight-line graph is drawn as evenly as possible through the plotted points, and on it two suitable points A and B are selected (Fig. 39).

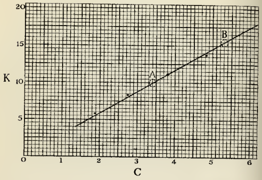


FIG. 39.

For the point A, $C = 3{\cdot}4$ and $K = 10$.
For the point B, $C = 5{\cdot}3$ and $K = 15{\cdot}5$.
These values satisfy the above law.

Then (1) $10 = 3{\cdot}4m + b$
(2) $15{\cdot}5 = 5{\cdot}3m + b$

By subtraction $5{\cdot}5 = 1{\cdot}9m$

$$\therefore \quad m = \frac{5{\cdot}5}{1{\cdot}9} = 2{\cdot}9$$

Substituting in (1) $10 = (3{\cdot}4 \times 2{\cdot}9) + b$
that is $10 = 9{\cdot}86 + b$
so that $b = 0{\cdot}14$
Hence the law is $\mathbf{K = 2{\cdot}9C + 0{\cdot}14}$.

EXERCISE VII

1. The air pressure on the front of an engine at different speeds is as follows:

Speed in m.p.h.	10	20	30	35	40	50	55
Pressure in lb wt per sq ft	0·3	1·2	2·7	3·675	4·8	7·5	9·075

Show by a graph the relation between speed and pressure. Find the pressure when the speed is 45 m.p.h., and the speed when the pressure is 3 lb wt per sq ft. (N.C.T.E.C.)

2. The pressures at different depths in a certain liquid are found to be as follows:

Depth in in.	0	4	8	12	14
Pressure in lb wt per sq in.	15	37·9	60·8	83·75	96·15

Show graphically the relation between pressure and depth. From the graph obtain—

(1) The pressure at a depth of 11 in.

(2) The depth at which the pressure is 43·5 lb per sq in. (N.C.T.E.C.)

3. The following table gives the tapping sizes for Whitworth threads up to 1 in. diameter. (The tapping size is the diameter of the hole drilled to permit the threading tap to enter.) Determine the law $T = aD + b$ connecting tapping size T with screw diameter D.

D in.	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
T in.	0·197	0·312	0·406	0·531	0·641	0·75	0·86

(Coventry.)

4. A comparison of degrees Centigrade (C) and degrees Fahrenheit (F) is given in the following table:

C . .	10	25	50	60	75	100
F . .	50	77	122	140	167	212

Draw a graph showing the relation between F and C, and determine its equation in the form

$$F = mC + n$$

where m and n are constants. From the graph determine:

(*a*) How many degrees F correspond to 30° C.
(*b*) How many degrees C correspond to 41° F.

(Coventry.)

5. The velocity of a body at intervals of 1 sec over a period of 7 sec is given by the following table:

t .	0	1	2	3	4	5	6	7
v .	0	5	18	38	62	78	81	83

Draw a graph of v against t, and from it find the distance travelled by the body during the 7 sec (this is given by the area under the graph) by a method other than counting squares. (Cannock.)

6. In an engine test the value of the indicated horse-power I, and the brake horse-power B, were as follows:

I .	1·197	4·17	5·85	7·96	8·91	10·77	14·68
B .	0	2·91	4·55	6·32	7·45	9·05	12·3

Verify by means of a graph that $B = aI + b$, and determine from your graph the values of a and b.

(Cannock.)

NOTE.—It is improbable that either I or B could be measured reliably to three or four significant figures—the actual figures quoted arise no doubt from slide-rule calculations. By plotting, erratic variations can be smoothed out, and values of a and b reliable to two or three significant figures obtained.

7. The length l in. of a helical spring when supporting a weight w lb is as follows:

w (lb) . .	2	5·1	10	15·4
l (in.) . .	5·8	7·05	9	11·2

Plot a graph, with l measured vertically, and find (i) the length of the spring when no weight is attached; (ii) the length when a weight of 7·8 lb is attached; (iii) the equation connecting l and w. (Sunderland.)

8. The amount of stretching l in. which takes place in a steel bar, when subjected to varying tensions T lb wt, is as follows:

Tension T (lb wt) .	140	400	500	720	850	1000
Amount of stretching l (in.) . . .	0·3	0·8	0·98	1·45	1·7	1·97

Draw a straight-line graph which appears to correspond most closely with these measurements. (Choose l as the vertical axis.)

Find the probable stretching when the tension is 300 lb wt, and also when it is 650 lb wt; and find the probable tension when the stretching is 0·45 in.

Find also the equation connecting l and T.

(Sunderland.)

9.

t (deg C.)	0	10	20	30	40	50	60
L (metres)	100	100·02	100·04	100·059	100·081	100·099	100·19

The table shows values of the length of a rod at various temperatures. Plot a graph of L against t (t horizontally) choosing suitable scales.

If the law is $L = a + bt$, find the value of a and b from the graph and hence the straight-line law connecting L and t. (Coventry.)

10. When current is taken from a primary cell the internal resistance of the cell causes a drop in the terminal voltage. During a test the following values of the output current I and the terminal voltage E were obtained:

I . .	0·1	0·3	0·5	0·7	0·9	1·0
E . .	1·44	1·32	1·2	1·08	0·96	0·9

Draw a graph showing the relation between I and E and determine the law connecting I and E in the form $E = V - rI$, where V and r are constants.

What would be the value of the terminal voltage when no current is taken from the cell? (Nuneaton.)

11. In a test on a steel bar the following values of load (W) and extension (E) were found. Some of the values are missing.

W . .	2	—	6·5	8·5	10	12
E . .	0·64	0·84	1·54	1·94	—	2·64

W and E are thought to be related by a law of the form $E = aW + b$. Show graphically that this is so and find the values of a and b.

Insert the missing values in the table. (E.M.E.U.)

12. Plot the points (7, 9) and (− 3, 5) and find the co-ordinates of the middle point of the line which joins them.

13. Draw the straight line which passes through the points (3, 2) and (− 2, 1) and find its intercepts on the y axis and on the x axis.

14. The following equations represent straight lines. Draw them and find the intercepts on the axes of y and x in each case.

(a) $y = -2x + 5$.
(b) $3y = 6x + 9$.
(c) $y - x = 2$.
(d) $2y = 9x + 6$.
(e) $y = -2{\cdot}4x - 7{\cdot}2$.

15. Solve the following simultaneous equations graphically by noting the point of intersection of the pair of straight lines in each case.

(a) $2y = -x + 12$, and $y = \frac{1}{3}x - \frac{2}{3}$.
(b) $y = 5x + 4$, and $y = -x + 2$.
(c) $4y - 4x = 12$, and $x = 2$.

16. Draw lines through the following pairs of points and determine the law connecting x and y in each case.

(a) (3, 5) and (− 5, − 2).
(b) (− I, I0) and (2, − 4).

17. The following table gives values of x and y which are connected by a law of the form $y = ax + b$.

Plot the corresponding points and draw a straight line to lie evenly amongst them, and from this line determine the values of a and b.

x	0	3	4	−1	−3	−5
y	1	2	2·3	0·7	0	−0·7

18. In certain experiments carried out with a machine, the effort E and the load W were found to have the values

as set out below. The law connecting E and W is of the form $E = aW + b$, where a and b are constants. Find this law by drawing the line which lies evenly between the points.

W . . .	30	40	60	70	80
E . . .	2·13	2·6	3·8	4·3	5·1

Work out the fraction $\frac{W}{E}$ for each pair of values. Add these quotients as a line to your table and plot them against W.

19. In a series of experiments to determine the friction F lb between two metallic surfaces when the load is W lb, the following results were obtained:

W . . .	3	5	7	10	12
F . . .	0·62	1·5	2·4	3·6	4·4

Assuming W and F to be connected by a law of the form $F = aW + b$, find this law by drawing the average straight line between the points.

20. The velocity v of a body at the end of an interval of t sec was found by a series of experiments to be as shown below:

t . . .	1	3	5	9	12
v . . .	10·6	11·75	13	15·5	17

If v and t are connected by a law of the form $v = u + at$, find u and a.

21. Tests carried out to determine the breaking stress (S) of rolled copper at different temperatures (t) gave the following results:

S	17·8	16·9	16·4	15·4	15·0	14·2
t (degrees) . .	60	210	300	410	500	600

Plot to as big a scale as the paper will allow S vertically and t horizontally. Write down the scales used. Now draw a straight line to lie evenly amongst the points obtained. From the diagram find the connection between S and t in the form $S = a + bt$, where a and b are numbers and determine the values of a and b. (U.E.I.)

22. In an experiment on a crane the load lifted (W lb) and the corresponding effort (E lb) required were found to be as under:

W	14	42	84	112
E	5·1	13·3	26·0	35·3

Plot to as large a scale as your paper will allow E vertically against W horizontally, draw a straight line to lie evenly among the points obtained and write down the scales used.

Using the diagram and assuming that E and W are connected by a law of the form $E = aW + b$, where a and b are numbers, find the values of a and b. (U.E.I.)

Referring to exercise 18 plot also a graph to show the relation between $\frac{W}{E}$ and W.

23. The volume (V) of a gas at various temperatures t is given by the following table.

t	10	20	30	40	50	60
V	95	98·5	101·8	105·3	108·7	112·2

Draw a graph showing the relation between V and t, and determine its equation in the form $V = at + b$, where a and b are constants.

What would be the volume when $t = 0$? (U.L.C.I.)

24. Set off on squared paper two axes of reference, OX horizontally and OY vertically.

Plot the point $(1, \frac{1}{2})$ and mark it P.

Through P draw lines PA, PB and PC the gradients of which are respectively 2, $1\frac{1}{4}$ and $\frac{1}{2}$. (U.E.I.)

25. (*a*) State which of the following functions will give straight line graphs: $\frac{2}{x}$; $5(x - 2)$; $\frac{x}{2}$; $1 - x$; $x^2 + 1$; $3 - \frac{1}{2}x$; $2x(x + 2)$.

(*b*) Draw straight lines through the point (0, 1) whose gradients are $\frac{1}{2}$, -3, $1{\cdot}5$, $-\frac{2}{3}$. (N.C.T.E.C.)

26. State which of the following functions will give straight-line graphs, and which will not: $(x + 1)(x + 2)$; $\frac{1}{3x}$; $2 - 5x$; $0{\cdot}5(x - 3)$; $x^2 + 2$; $1{\cdot}7x$; $\frac{x + 3}{5}$.

(N.C.T.E.C.)

CHAPTER 8

INDICES—LOGARITHMS

1. The Index Notation

We have seen in Chapter 3 that a^4 is a short way devised in Algebra for writing $a \times a \times a \times a$. The figure 4 is called an index, and indicates the number of factors.

Generally a^n means $a \times a \times a \times a \ldots$ to n factors and *a^n is called the n^{th} power of a.*

NOTE.—a represents any number.

2. Laws of Indices

The laws regulating the use of Indices have been briefly touched on in Chapter 3. We must now consider them more fully.

(1) Law of Multiplication

We have already seen that since

a^5 means the product of 5 a's

and a^3 ,, ,, ,, 3 a's,

then $a^5 \times a^3$ must mean the product of $(5 + 3)a$'s.

i.e.
$$a^5 \times a^3 = a^{5+3} = a^8$$

Clearly this will always be true whatever powers are taken, provided they are positive integers.

$\therefore$ If m and n be positive integers

$$a^m \times a^n = a^{m+n}$$

The law is true, obviously, for more than two factors.

For example $$a^2 \times a^5 \times a^4 = a^{2+5+4}$$
$$= a^{11}$$

(2) Law of Division

It has previously been shown (p. 53) that if we want to divide a^5 by a^3

since $a^5 = a \times a \times a \times a \times a$
and $a^3 = a \times a \times a$

on division the three factors of a^3 cancel with three of the five factors of a^5

$\therefore$ there are left $(5 - 3)$ factors, each of them a.

$$\therefore \quad a^5 \div a^3 = a^{5-3}$$
$$= a^2$$

It will be seen that the same method may be applied for any powers of a, provided that the index of the divisor be less than that of the dividend.

Hence in general if m and n be positive integers

$$a^m \div a^n = a^{m-n}$$

(3) Law of Powers

Suppose we require to find the value of $(a^5)^3$. By this we mean that we require the third power of a^5.

By the definition of an index $(a^5)^3$ means $a^5 \times a^5 \times a^5$.

But by the law of multiplication

$$a^5 \times a^5 \times a^5 = a^{5+5+5}$$
$$= a^{5\times 3}$$
$$= a^{15}$$

Hence $$(a^5)^3 = a^{5\times 3}$$
$$= a^{15}$$

The same kind of reasoning will follow in other cases, and so generally, if m and n are positive integers

$$(a^m)^n = a^{m\times n}$$

The student should now work Exercise VIII, Section A, p. 187.

3. Extension of the Meaning of an Index

The student will readily understand how useful and important indices are in Algebra. He will note that so far they have been restricted to positive whole numbers only, and the meaning given to such a quantity as a^n is unintelligible except on the supposition that n is a positive integer. But we will now consider the possibility of extending the uses of indices so that they can have any value.

The student may already have noticed one instance which will be among those we sball consider in detail later. If we divide a^3 by a^5 and write this down in the form $\dfrac{a \times a \times a}{a \times a \times a \times a \times a}$, we obtain on cancelling $\dfrac{1}{a \times a}$ or $\dfrac{1}{a^2}$.

If a^3 be divided by a^5 according to rule we have

$$a^3 \div a^5 = a^{3-5}$$
$$= a^{-2}$$

We are thus left with a negative index. But the working above shows that the result of the division of a^3 by a^5 is $\dfrac{1}{a^2}$.

Consequently it appears that a^{-2} means the same thing as $\dfrac{1}{a^2}$, or the reciprocal of a^2.

Thus it appears that a meaning can be given to a^{-2} by application of the rules developed for the case when the index is a positive whole number. We are therefore led to consider what meanings can be given in all those cases in which the index is not a positive integer. In seeking these

meanings of an index there is one fundamental principle which will always guide us, viz.: *Every index must obey the laws of indices as discovered for positive integers.* In other words, we will assume that the laws of indices, as discovered above, are true in all cases.

4. Fractional Indices

We will begin with the simple case of $a^{\frac{1}{2}}$. Since, by the above principle, it must conform to the laws of Indices, then, applying the law of multiplication

$$\begin{aligned} a^{\frac{1}{2}} \times a^{\frac{1}{2}} &= a^{\frac{1}{2}+\frac{1}{2}} \\ &= a^1 \text{ or } a \end{aligned}$$

$\therefore$ $a^{\frac{1}{2}}$ must be such a quantity that, on being multiplied by itself, the result is a.

$\therefore$ *$a^{\frac{1}{2}}$ must be defined as the square root of a*

$$\text{or } a^{\frac{1}{2}} = \sqrt{a}$$

Similarly

$$\begin{aligned} a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} &= a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= a \end{aligned}$$

(First law of indices.)

$\therefore$ *$a^{\frac{1}{3}}$ must de defined as the cube root of a.*

The same argument may be applied in other cases, and so generally

$$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

To find a meaning for $a^{\frac{2}{3}}$

Applying the first law of indices

$$\begin{aligned} a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} &= a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} \\ &= a^2 \end{aligned}$$

$\therefore$ *$a^{\frac{2}{3}}$ must be the cube root of a^2*

or $\qquad a^{\frac{2}{3}} = \sqrt[3]{a^2}$

Similarly $\qquad a^{\frac{3}{4}} = \sqrt[4]{a^3}$

and generally $\qquad a^{\frac{m}{n}} = \sqrt[n]{a^m}$

The student will note that decimal indices can be reduced to vulgar fractions and defined accordingly.

Thus $\qquad$ $$\begin{aligned} a^{0\cdot 25} &= a^{\frac{1}{4}} \\ &= \sqrt[4]{a} \end{aligned}$$

5. To find a meaning for a^0

$$a^n \div a^n = 1$$

But, using the law of division for Indices,

$$\begin{aligned} a^n \div a^n &= a^{n-n} \\ &= a^0 \\ \therefore \quad a^0 &= 1 \end{aligned}$$

It should be noted that a represents any number. This result therefore is independent of the value of a.

6. Negative Indices

To find a meaning for a^{-n}

$$\begin{aligned} a^{-n} \times a^n &= a^{-n+n} \text{ (first law of indices)} \\ &= a^0 \\ &= 1 \quad \text{(shown above)} \end{aligned}$$

Dividing by a^n

$$a^{-n} = \frac{1}{a^n}$$

We may therefore define a^{-n} as the reciprocal of a^n.

Examples.

$$a^{-1} = \frac{1}{a}$$

$$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$$

$$\frac{1}{a^{-2}} = a^2$$

or generally $\qquad \dfrac{1}{a^{-n}} = a^n$

Example 1. *If* $\sqrt{2} = 1{\cdot}414$ . . . *find the value of* $2^{\frac{3}{2}}$.

$$2^{\frac{3}{2}} = 2^{1+\frac{1}{2}} = 2 \times 2^{\frac{1}{2}} \text{ (first law of indices)}$$
$$= 2 \times \sqrt{2}$$
$$= 2 \times 1{\cdot}414 = 2{\cdot}828$$

Example 2. *If* $\sqrt{10} = 3{\cdot}1623$ . . . *find the value of* $10^{-\frac{1}{2}}$.

$$10^{-\frac{1}{2}} = \frac{1}{10^{\frac{1}{2}}} = \frac{1}{\sqrt{10}}$$
$$= \frac{\sqrt{10}}{10} = \frac{3{\cdot}1623}{10}$$
$$= 0{\cdot}31623$$

The student should now work Exercise VIII, Section, B, p. 188.

7. A System of Logarithms

These extensions of the use of indices to all real values are of great practical importance. For example, they enable us to carry out, easily and accurately, calculations which without them would be almost impossible or very laborious. The fundamental idea may be illustrated by a very simple example.

Suppose we want to find the value of 16×64. The ordinary method of multiplication could be replaced by the following:

$$16 = 2^4$$
$$64 = 2^6$$
$$\therefore \quad 16 \times 64 = 2^4 \times 2^6$$
$$= 2^{10}$$

Now, if we had a table of powers of 2, we could look up the value of 2^{10}, which is 1024, and thus obtain the answer. Thus the process of multiplying 16 and 64 is replaced by that of *adding the indices* 4 and 6. This would be of very little practical value if we were confined to positive integral

indices. We could then only deal with a few special cases. But the extension of indices to include all kinds of numbers will enable us, as we shall see, to perform complicated arithmetical operations. The first essential is a table of powers. In a simple way we could construct this graphically as follows

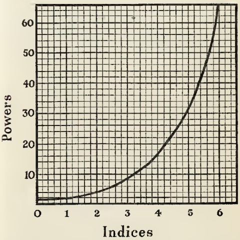


FIG. 40.

Fig. 40 represents a curve of 2^x, *i.e.* of powers of 2, between the values $x = 0$, and $x = 6$. A smooth curve can be drawn by taking the values of 2^1, 2^2, 2^3, etc. It should be noted that the smoothness of the curve itself suggests that there are values of 2^x for values of x other than the integral values 1, 2, 3 . . . By the application of the rules

of indices which we have discovered we can now find other values. Thus

$$2^0 = 1$$
$$2^{\frac{1}{2}} = \sqrt{2} = 1{\cdot}414$$
$$2^{\frac{3}{2}} = 2 \times 2^{\frac{1}{2}} = 2 \times 1{\cdot}414 = 2{\cdot}83$$

and so on.

These values may be plotted, and as they all lie on the smooth curve, we have a confirmation of the reasoning by which the meanings of fractional indices were obtained.

NOTE.—In order to obtain the results given below the student will need to draw the curve on a much larger scale than can be printed here.

Let us now use the curve to find the value of

$$6{\cdot}5 \times 8{\cdot}8$$

From the curve $6{\cdot}5 = 2^{2\cdot7}$ (roughly)
and $8{\cdot}8 = 2^{3\cdot14}$

$$\therefore\ 6{\cdot}5 \times 8{\cdot}8 = 2^{2\cdot7} \times 2^{3\cdot14}$$
$$= 2^{2\cdot7+3\cdot14} \text{ (first law of indices)}$$
$$= 2^{5\cdot84}$$

From the curve we find that roughly

$$2^{5\cdot84} = 57$$
$$\therefore\ 6{\cdot}5 \times 8{\cdot}8 = 57 \text{ approximately.}$$

Although interesting as illustrating the principles involved, the above method has little practical value, since we must depend, for the values required, on a curve which is necessarily limited in size and not sufficiently accurate. To use the method effectively we need a table from which we can obtain, to any required degree of accuracy, values such as those which are required in the above and similar problems. For such a table we must have a number called the **base,** just as we selected 2 above, and then the table must give the index showing the power which any given

number is of that base. For practical purposes, as will be seen, the most suitable base is 10 and the indices which express numbers as powers of 10 can be calculated by methods of advanced mathematics. Such a table of indices is called a **table of Logarithms.** We can therefore define a logarithm as follows.

Definition. The logarithm of a number to a given base is the index of the power to which the base must be raised to produce the number.

For example, we know that $168{\cdot}3 = 10^{2\cdot2261}$.

Then, by the above definition, 2·2261 is the logarithm of 168·3 to the base 10.

Similarly, since $32 = 2^5$, 5 is the logarithm of 32 to base 2.

Notation of Logarithms.

When we wish to express the logarithm of a number with reference to a given base, we use the following notation.

Since, as we have seen above

$$168{\cdot}3 = 10^{2\cdot2261}$$

in which 2·2261 is the index or logarithm and 10 is the base, we write this connection thus:

$$2{\cdot}2261 = \log_{10} 168{\cdot}3$$

The base, 10, is indicated by writing the 10 as shown.

Similarly since

$$1{,}000 = 10^3,\ 3 = \log_{10} 1{,}000$$

also since $32 = 2^5,\ 5 = \log_2 32$

Both of these forms are used, and the student should practise himself in changing from one form to the other.

8. Characteristic

The integral or whole number part of a logarithm is called the characteristic. This can always be determined

by inspection when logarithms are calculated to base 10, as will be seen from the following considerations:

$$\begin{array}{llll} \text{Since} & 10^0 = 1, & \log_{10} 1 & = 0 \\ & 10^1 = 10, & \log_{10} 10 & = 1 \\ & 10^2 = 100, & \log_{10} 100 & = 2 \\ & 10^3 = 1{,}000, & \log_{10} 1{,}000 & = 3 \\ & 10^4 = 10{,}000, & \log_{10} 10{,}000 & = 4 \end{array}$$

and so on.

From these results we see that,

for numbers between 1 and 10 the characteristic is 0
" " " 10 " 100 " " " 1
" " " 100 " 1,000 " " " 2
" " " 1,000 " 10,000 " " " 3
and so on.

It is evident that **the characteristic is always one less than the number of digits in the whole number part of the number.**

Thus in $\log_{10} 3758{\cdot}7$ the characteristic is 3
$\log_{10} 375{\cdot}87$ " " " 2
$\log_{10} 37{\cdot}587$ " " " 1

Thus the characteristic may always be determined by inspection, and consequently is not given in the tables. This is one advantage of having 10 for a base.

9. Mantissa

The decimal part of a logarithm is called the mantissa.

In general the mantissa can be calculated to any required number of figures, by the use of higher mathematics. In most tables, such as those given in this volume, the mantissa is stated to *four* places of decimals. In *Chambers' "Book of Tables"* it is given to seven places of decimals.

The mantissa alone is given in the tables, and the following example will show that this is sufficient:

$$\begin{aligned} \log_{10} 168{\cdot}3 &= 2{\cdot}2261 \\ \therefore\quad 168{\cdot}3 &= 10^{2{\cdot}2261} \\ \therefore\quad 168{\cdot}3 \div 10 &= 10^{2{\cdot}2261} \div 10^1 \\ \therefore\quad 16{\cdot}83 &= 10^{2{\cdot}2261-1} \text{ (second law of indices)} \\ &= 10^{1{\cdot}2261} \\ \therefore\quad \log_{10} 16{\cdot}83 &= 1{\cdot}2261 \\ \text{Similarly } \log_{10} 1{\cdot}683 &= 0{\cdot}2261 \\ \text{and}\quad \log_{10} 1683 &= 3{\cdot}2261 \end{aligned}$$

Thus, if a number is multiplied or divided by a power of 10, the characteristic of the logarithm of the result is changed, but **the mantissa remains unaltered.** This may be expressed as follows:

Numbers having the same set of significant figures have the same mantissa in their logarithms.

10. To read a Table of Logarithms

With the use of the above rules relating to the characteristic and mantissa of logarithms, the student should have no difficulty in reading a table of logarithms.

Below is a portion of such a table, giving the logarithms of numbers between 31 and 36.

No	Log.	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
(1)	(2)																		

The figures in column 1 in the complete table are the numbers from 1 to 99. The corresponding number in column 2 is the mantissa of the logarithm. As previously stated, the characteristic is not given, but can be written

down by inspection. Thus $\log_{10} 31 = 1{\cdot}4914$, $\log_{10} 310 = 2{\cdot}4914$, etc. If the number has a *third significant figure*, the mantissa will be found in the appropriate column of the next nine columns.

Thus $\quad \log_{10} 31{\cdot}1 = 1{\cdot}4928,$
$\quad\quad \log_{10} 31{\cdot}2 = 1{\cdot}4942,$ and so on.

If the number has a *fourth significant figure* space does not allow us to give the whole of the mantissa. But the next nine columns of what are called "mean differences" give us for every fourth significant figure a number which must be added to the mantissa already found for the first three significant figures. Thus if we want $\log_{10} 31{\cdot}67$, the mantissa for the first three significant figures 316 is 0·4997. For the fourth significant figure 7 we find in the appropriate column of mean differences the number 10. This is added to 0·4997 and so we obtain for the mantissa 5007.

$$\therefore \quad \log_{10} 31{\cdot}67 = 1{\cdot}5007$$

Anti-logarithms.

The student is usually provided with a table of anti-logarithms which contains the *numbers corresponding to given logarithms*. These could be found from a table of logarithms, but it is quicker and easier to use the anti-logarithms.

The tables are similar in their use to those for logarithms, but we must remember—

(1) that the mantissa of the log only is used in the table;

(2) when the significant figures of the number have been obtained, the student must proceed to fix the decimal point in them by using the rules which we have considered for the characteristic.

Example. *Find the number whose logarithm is 2·3714.*

First using the mantissa—viz. 0·3714—we find from the anti-logarithm table that the number corresponding is given as **2352**. These are the first four significant figures of the number required.

Since the characteristic is 2, the number must lie between 100 and 1000 (see p. 176) and therefore it must have 3 significant figures in the integral part.

$\therefore$ The number is **235·2.**

NOTE.—As the log tables which will be usually employed by the beginner are all calculated to base 10, the base in further work will be omitted when writing down logarithms. Thus we shall write $\log 235{\cdot}2 = 2{\cdot}3714$, the base 10 being understood.

The student is now advised to work Exercise VIII, Section C.

11. Rules for the Use of Logarithms

In using logarithms for calculations we must be guided by the laws which govern their use. Since logarithms are indices, these laws must be the same in principle as those of indices. These rules are given below; formal proofs of them will be given in Vol. II of this series.

(1) Logarithm of a Product

The logarithm of the product of two or more numbers is equal to the sum of the logarithms of these numbers (see first law of indices).

Thus if p and q be any numbers

$$\log (p \times q) = \log p + \log q$$

(2) Logarithm of a Quotient

The logarithm of p divided by q is equal to the logarithm of p diminished by the logarithm of q (see second law of indices).

Thus $\quad \log (p \div q) = \log p - \log q$

(3) Logarithm of a power

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power (see third law of indices).

Thus $$\log a^n = n \log a$$

(4) Logarithm of a Root

This is a special case of the above

Thus
$$\log \sqrt[n]{a} = \log a^{\frac{1}{n}}$$
$$= \frac{1}{n} \log a$$

12. Examples of the Use of Logarithms

Example 1. *Find the value of* $57{\cdot}86 \times 4{\cdot}385$.

First method.—Using the *method of indices*, from the tables we have:

$$57{\cdot}86 = 10^{1{\cdot}7624}$$
and
$$4{\cdot}385 = 10^{0{\cdot}6420}$$
$$\therefore\ 57{\cdot}86 \times 4{\cdot}385 = 10^{1{\cdot}7624} \times 10^{0{\cdot}6420}$$
$$= 10^{1{\cdot}7624 + 0{\cdot}6420}$$
$$= 10^{2{\cdot}4044}$$
$$= 253{\cdot}7$$

Second method.—Using the logarithm notation

Let $x = 57{\cdot}86 \times 4{\cdot}385$

Then $\log x = \log 57{\cdot}86 + \log 4{\cdot}385$

$= 1{\cdot}7624 + 0{\cdot}6420$

$= 2{\cdot}4044$

$= \log 253{\cdot}7$

$\therefore\ x = 253{\cdot}7$

No.	log.
57·86	1·7624
4·385	0·6420
253·7	2·4044

NOTES.—(1) First method is given to illustrate the dependence on the laws of indices, but in general the student will use the second method.

(2) The student should remember that the logs in the tables are correct to four significant figures only. Consequently he cannot be sure of four significant figures in the answer (see Chapter 1, p. 19). It would be more correct to give the above answer as 254, correct to three significant figures.

(3) The student is advised to adopt some systematic way of arranging the actual operations with logarithms. Such a method is shown on the right of the setting out of the reasoning.

Example 2. *Find the value of*

$$\frac{5{\cdot}672 \times 18{\cdot}94}{1{\cdot}758}$$

Using the second method given above.

Let $x = \dfrac{5{\cdot}672 \times 18{\cdot}94}{1{\cdot}758}$

$\therefore\ \log x = \log 5{\cdot}672 + \log 18{\cdot}94 - \log 1{\cdot}758$

$= 0{\cdot}7538 + 1{\cdot}2774 - 0{\cdot}2450$

$= 1{\cdot}7862$

$= \log 61{\cdot}12$

$\therefore\ x = 61{\cdot}12$

or $x = 61{\cdot}1$ (to three significant figures)

No.	log.
5·672	0·7538
18·94	1·2774
	2·0312
1·758	0·2450
61·12	1·7862

Example 3. *Find the fifth root of* $721{\cdot}8$.

Let $x = \sqrt[5]{721{\cdot}8}$

$= (721{\cdot}8)^{\frac{1}{5}}$

Then $\log x = \frac{1}{5} \log 721{\cdot}8$

$= \frac{1}{5}(2{\cdot}8584)$

$= 0{\cdot}5717$

$\therefore\ x = 3{\cdot}730$

Example 4. *If* $c = \sqrt{a^2 - b^2}$ *find c when* $a = 7{\cdot}83$ *and* $b = 2{\cdot}85$.

NOTE.—In this example it should be noted that $a^2 - b^2$ can be factorised. Then we shall have the square root of a product and logs can readily be applied.

$$c = \sqrt{a^2 - b^2}$$
$$= \sqrt{(a+b)(a-b)}$$
$$= \sqrt{10{\cdot}68 \times 4{\cdot}98}$$
$$\therefore \quad \log c = \tfrac{1}{2}(\log 10{\cdot}68 + \log 4{\cdot}98)$$
$$= \log 7{\cdot}293 \text{ (side working)}$$
$$\therefore \quad c = 7{\cdot}293$$
or $\quad c = 7{\cdot}29$ (to three significant figures)

No.	log.
10·68	1·0286
4·98	0·6972
÷ by 2	1·7258
7·293	0·8629

It should be carefully noted that logarithms could only be used to evaluate the expression $\sqrt{a^2 - b^2}$ as a whole, because $a^2 - b^2$ could readily be factorised. In general, if an expression to be evaluated contains terms separated by a sign + or −, logarithms can be used only to evaluate the separate terms which must then be added or subtracted by ordinary arithmetic. The temptation to seek to carry out additions or subtractions with the aid of logarithms must be resisted.

Example 5. *If* $c = \sqrt{a^2 + b^2}$, *find c when* $a = 7{\cdot}83$ *and* $b = 2{\cdot}85$.

We must first evaluate separately a^2 and b^2, and since a and b each have three significant figures it may pay us to use logarithms to do this.

No.	log.	No.	log.
7·83	0·8938	2·85	0·4548
	2		2
61·32	1·7876	8·121	0·9096

By ordinary addition $61{\cdot}32 + 8{\cdot}121 = 69{\cdot}441$.

We now have to find the square root of 69·44 for which again we can use logarithms.

No.	log.
69·44	1·8416
8·333	0·9208, dividing by 2

So the result required is

$$c = 8{\cdot}33 \text{ to three significant figures.}$$

The student should now work Exercise VIII, Section D.

13. Logarithms of Numbers between 0 and 1

On p. 176, we gave examples of powers of 10 when the index is a positive integer. We will now consider cases in which the indices are negative. In doing so we must be guided by the meanings of such indices as found on p. 171.

$$\begin{array}{llll}
\text{Thus} & 10^1 = 10 & \therefore & \log_{10} 10 = 1 \\
& 10^0 = 1 & \therefore & \log_{10} 1 = 0 \\
& 10^{-1} = \frac{1}{10} = 0{\cdot}1 & \therefore & \log_{10} 0{\cdot}1 = -1 \\
& 10^{-2} = \dfrac{1}{10^2} = 0{\cdot}01 & \therefore & \log_{10} 0{\cdot}01 = -2 \\
& 10^{-3} = \dfrac{1}{10^3} = 0{\cdot}001 & \therefore & \log_{10} 0{\cdot}001 = -3
\end{array}$$
etc.

From these results we may deduce that—

The logarithms of numbers between 0 *and* 1 *are always negative.*

We have seen (p. 177) that if a number be divided by 10, we obtain the log of the result by subtracting 1.

$$\begin{array}{ll}
\text{Thus if} & \log 49{\cdot}8 = 1{\cdot}6972 \\
& \log 4{\cdot}98 = 0{\cdot}6972 \\
& \log 0{\cdot}498 = 0{\cdot}6972 - 1 \\
& \log 0{\cdot}0498 = 0{\cdot}6972 - 2 \\
& \log 0{\cdot}00498 = 0{\cdot}6972 - 3
\end{array}$$

From the above $\log 0{\cdot}498 = 0{\cdot}6972 - 1$
$$= -0{\cdot}3028$$

Now, in the logs of numbers greater than unity, the mantissa remains the same when the numbers are multiplied or divided by powers of 10 (see p. 177), *i.e.* with the same significant figures we have the same mantissa.

It would clearly be a great advantage if we could find a system which would enable us to use this rule for numbers less than unity, and so avoid, for example, having to write

$$\log 0{\cdot}498 \text{ as } -0{\cdot}3028$$

This can be done by retaining the characteristic as negative instead of carrying out the subtraction shown above. But to write $\log 0{\cdot}498$ as $0{\cdot}6972 - 1$ would be awkward. Accordingly we adopt the notation $\bar{1}{\cdot}6972$, writing the minus sign above the characteristic.

It is very important to remember that

$$\bar{1}{\cdot}6972 = -1 + 0{\cdot}6972$$

Thus in logarithms written in this way **the characteristic is negative and the mantissa is positive.**

With this notation $\log 0{\cdot}0498 = \bar{2}{\cdot}6972$
$\log 0{\cdot}00498 = \bar{3}{\cdot}6972$
$\log 0{\cdot}000498 = \bar{4}{\cdot}6972$ etc.

NOTE.—The student should note that *the negative characteristic is numerically one more than the number of zeros after the decimal point.*

Example 1. *From the tables find the logs of* 0·3185, 0·03185 and 0·003185.

Using the portion of the tables on p. 177, we see that the mantissa for 0·3185 will be 0·5031.

Also the characteristic is -1.

$\therefore$ $\log 0{\cdot}3185 = \bar{1}{\cdot}5031$
Similarly $\log 0{\cdot}03185 = \bar{2}{\cdot}5031$
and $\log 0{\cdot}003185 = \bar{3}{\cdot}5031$

Example 2. *Find the number whose log is* $\bar{3}{\cdot}5416$.

From the tables we find that significant figures of the number whose mantissa is 5416 are 3480. As the characteristic is -3, there will be two zeros after the decimal point.

$\therefore$ the number is 0·003480

The student should now work Exercise VIII, Section E, Nos. 1–3.

15. Operations with Logarithms which are Negative

Care is needed in dealing with the logarithms of numbers which lie between 0 and 1, since they are negative and, as shown above, are written with the characteristic negative and the mantissa positive.

A few examples will show the methods of working.

Example 1. *Find the sum of the logarithms:*

$\bar{1}{\cdot}6173$, $\bar{2}{\cdot}3415$, $\bar{1}{\cdot}6493$, $0{\cdot}7374$

Arranging thus

$$\begin{array}{r} \bar{1}{\cdot}6173 \\ \bar{2}{\cdot}3415 \\ \bar{1}{\cdot}6493 \\ 0{\cdot}7374 \\ \hline \bar{2}{\cdot}3455 \end{array}$$

The point to be specially remembered is that the 2 which is carried forward from the addition of the mantissæ is positive, since they are positive. Consequently the addition of the characteristics becomes

$$-1-2-1+0+2=-2$$

Example 2. *From the logarithm* $\bar{1}{\cdot}6175$ *subtract the log* $\bar{3}{\cdot}8463$.

$$\begin{array}{r} \bar{1}{\cdot}6175 \\ \bar{3}{\cdot}8463 \\ \hline 1{\cdot}7712 \end{array}$$

Here in " borrowing " to subtract the 8 from the 6, the -1 in the top line becomes -2, consequently on subtracting the characteristics we have

$$-2-(-3)=-2+3=+1$$

Example 3. *Multiply* $\bar{2}\cdot 8763$ *by* 3.

$$\begin{array}{r} \bar{2}\cdot 8763 \\ 3 \\ \hline \bar{4}\cdot 6289 \end{array}$$

From the multiplication of the mantissa, 2 is carried forward. But this is positive and as $(-2)\times 3=-6$, the characteristic becomes $-6+2=-4$.

Example 4. *Multiply* $\bar{1}\cdot 8738\times 1\cdot 3$.

In a case of this kind it is better to multiply the characteristic and mantissa separately and add the results.

Thus $\quad 0\cdot 8738\times 1\cdot 3=1\cdot 13594$

$\quad -1\times 1\cdot 3=-1\cdot 3$

$-1\cdot 3$ is wholly negative and so we change it to $\bar{2}\cdot 7$, to make the mantissa positive.

Then the product is the sum of

$$\begin{array}{l} 1\cdot 13594 \\ \bar{2}\cdot 7 \\ \hline \bar{1}\cdot 83594 \end{array}$$

or $\quad \bar{1}\cdot 8359$ *approx.*

Example 5. *Divide* $\bar{5}\cdot 3716$ *by* 3.

Here the difficulty is that on dividing $\bar{5}$ by 3 there is a remainder 2 which is negative, and cannot therefore be carried on to the positive mantissa. To get over the difficulty we write:

$$-5=-6+1$$

or the log as $\quad -6+1\cdot 3716$

Then the division of the -6 gives us -2 and the division of the positive part $1\cdot 3716$ gives $0\cdot 4572$, which is positive. Thus the complete quotient is $\bar{2}\cdot 4572$. The work might be arranged thus:

$$\begin{array}{r} 3)\bar{6}+1\cdot 3716 \\ \hline \bar{2}+0\cdot 4572 \\ \hline =\bar{2}\cdot 4572 \end{array}$$

The student should now work Exercise VIII, Section E, Nos. 4–8, followed by Sections F and G.

EXERCISE VIII

Section A. Laws of Indices

1. Write down the values of:

(1) $a^4\times a^6$.	(4) $\frac{1}{2}x\times\frac{1}{4}x^7\times\frac{3}{4}x^2$.
(2) $b^7\times b^5$.	(5) $2^3\times 2^4$.
(3) $x^3\times x^4\times x^5$.	(6) $3\times 3^2\times 3^4$.

2. Write down the values of:

(1) $a^7\div a^3$.	(3) $x^{16}\div x^4$.
(2) $c^{10}\div c^5$.	(4) $2^{10}\div 2^4$.

3. Find the values of:

(1) $x^7\times x^4\div x^5$.	(3) $\dfrac{a^7}{a^5}\times\dfrac{a}{a^4}$.
(2) $a^6\times a^5\div a^8$.	(4) $\dfrac{x^6\times x^4}{x^3}$.

4. Find the values of:

(1) $(a^7)^2$.	(5) $(10^2)^3$.
(2) $(x^4)^3$.	(6) $(3a^2)^3$.
(3) $(2b^4)^4$.	(7) $(\frac{1}{2}x^4)^5$.
(4) $(2^4)^2$.	(8) $(3^3)^3$.

Section B. Indices

Where necessary in the following take $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{10} = 3{\cdot}162$, each correct to three places of decimals.

1. Write down the meanings of:

$$3^{\frac{1}{2}},\ 4^{-1},\ 3a^{-2},\ 1000^{0},\ 2^{-\frac{1}{2}},\ \frac{1}{3^{-1}},\ \frac{3}{a^{-2}},\ 4^{\frac{3}{2}},\ 10^{-3}.$$

2. Find the values of:

(1) $2^2 \times 2^{\frac{1}{2}}$.	(4) $a^{\frac{1}{3}} \times a^{\frac{2}{3}}$.
(2) $3 \times 3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$.	(5) $2^{\frac{3}{2}}$.
(3) $10^{\frac{1}{2}} \div 10^{\frac{3}{2}}$.	(6) $10^{\frac{5}{2}}$.

3. Find the values of:

(1) $8^{\frac{2}{3}}$.	(4) $(5^{-3})^2$.
(2) $25^{\frac{3}{2}}$.	(5) $\dfrac{2}{2^{-3}}$.
(3) $(10^2)^{\frac{3}{2}}$.	(6) $(1000)^{\frac{1}{3}}$.

4. Find the values of:

(1) $(\frac{1}{2})^{-2}$.	(4) $(36)^{-0{\cdot}5}$.
(2) $(\frac{2}{3})^{-3}$.	(5) $(4)^{1{\cdot}5}$.
(3) $(16)^{0{\cdot}5}$.	(6) $(\frac{1}{4})^{2{\cdot}5}$.

5. Find the value of $a^4 \times a^{-2} \times a^{\frac{1}{2}}$ when $a = 2$.

6. Write down the simplest form of:

(1) $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$.	(2) $10^3 \times 10^{-\frac{1}{2}}$.

7. Find the values of:

(1) $32^{0{\cdot}5}$.	(4) $2^{0{\cdot}25}$.
(2) $8^{1{\cdot}5}$.	(5) $(16)^{-0{\cdot}25}$.
(3) $3^{2{\cdot}5}$.	(6) $(0{\cdot}6)^{-2}$.

8. If $10^{\frac{1}{3}} = 2{\cdot}154$ . . . to three places, find $10^{\frac{2}{3}}$ to two places.

9. Find the value of $\left(\dfrac{2^2}{3^3}\right)^{-1}$.

Section C. Log Tables

1. Write down the characteristics of the logarithms of the following numbers:

15, 1500, 31,672, 597, 8, 800,000
51·63, 3874·5, 2·615, 325·4.

2. Read from the tables the logarithms of the following numbers:

(1) 5, 50, 500, 50,000.
(2) 4·7, 470, 47,000.
(3) 52·8, 5·28, 528.
(4) 947·8, 9·478, 94,780.
(5) 5·738, 96·42, 6972.

3. Find, from the tables, the numbers of which the following are the logarithms:

(1) 2·65, 4·65, 1·65.
(2) 1·943, 3·943, 0·943.
(3) 0·6734, 2·6734, 5·6734.
(4) 3·4196, 0·7184, 2·0568.

Section D. Logarithmic Calculations

Use logarithms to find the values of the following:

1. $23{\cdot}4 \times 14{\cdot}73$.
2. $43{\cdot}97 \times 6{\cdot}284$.
3. $987{\cdot}4 \times 1{\cdot}415$.
4. $42{\cdot}7 \times 9{\cdot}746 \times 14{\cdot}36$.
5. $28{\cdot}63 \div 11{\cdot}95$.
6. $43{\cdot}97 \div 6{\cdot}284$.
7. $23{\cdot}4 \div 14{\cdot}73$.
8. $927{\cdot}8 \div 4{\cdot}165$.
9. $94{\cdot}76 \times 4{\cdot}195 \div 27{\cdot}94$.
10. $\dfrac{15{\cdot}36 \times 9{\cdot}47 \times 11{\cdot}48}{5{\cdot}632 \times 21{\cdot}85}$.
11. $(9{\cdot}478)^3$.
12. $(51{\cdot}47)^2$.
13. $(1{\cdot}257)^5$.
14. $(15{\cdot}23)^2 \times 3{\cdot}142$.
15. $(5{\cdot}98)^2 \div 16{\cdot}47$.
16. $\dfrac{(91{\cdot}5)^2}{4{\cdot}73 \times 16{\cdot}92}$.
17. $\dfrac{(8{\cdot}97)^2 \times (1{\cdot}059)^3}{57{\cdot}7}$.
18. $\dfrac{4798}{(56{\cdot}2)^2 \div (9{\cdot}814)^3}$.
19. $\sqrt[4]{3{\cdot}417}$.
20. $\sqrt[3]{4{\cdot}872}$.
21. $\sqrt[3]{1{\cdot}625^2 \times 4{\cdot}738}$.
22. $\sqrt[5]{61{\cdot}5 \times 2{\cdot}73}$.

23. If $\pi r^2 = 78{\cdot}6$ find r when $\pi = 3{\cdot}142$.

24. If $\frac{4}{3}\pi r^3 = 15{\cdot}5$, find r when $\pi = 3{\cdot}142$.

25. Find the difference between the areas of two squares whose sides are 9·74 in. and 5·66 in. (see p. 73 § 4).

26. If $M = PR^n$, find M when $P = 200$, $R = 1{\cdot}05$, $n = 20$.

Section E. Negative Logarithms

1. Write down the logarithms of:

(1) 2·798, 0·2798, 0·02798.
(2) 4·264, 0·4264, 0·004264.
(3) 0·009783, 0·0009783, 0·9783.
(4) 0·06451, 0·6451, 0·0006451.

2. Write down the logarithms of:

(1) 0·05986. (4) 0·00009275.
(2) 0·000473. (5) 0·5673.
(3) 0·007963. (6) 0·07986.

3. Find the numbers whose logarithms are:

(1) $\bar{1}{\cdot}3342$. (4) $\bar{4}{\cdot}6437$.
(2) $\bar{3}{\cdot}8724$. (5) $\bar{1}{\cdot}7738$.
(3) $\bar{2}{\cdot}4871$. (6) $\bar{8}{\cdot}3948$.

4. Add together the following logarithms:

(1) $\bar{2}{\cdot}5178 + 1{\cdot}9438 + 0{\cdot}6138 + \bar{5}{\cdot}5283$.
(2) $3{\cdot}2165 + \bar{3}{\cdot}5189 + \bar{1}{\cdot}3297 + \bar{2}{\cdot}6475$.

5. Find the values of:

(1) $4{\cdot}2183 - 5{\cdot}6257$. (3) $\bar{1}{\cdot}6472 - \bar{1}{\cdot}9875$.
(2) $0{\cdot}3987 - \bar{1}{\cdot}5724$. (4) $\bar{2}{\cdot}1085 - \bar{5}{\cdot}6271$.

6. Find the values of:

(1) $\bar{1}{\cdot}8732 \times 2$. (4) $\bar{1}{\cdot}5782 \times 1{\cdot}5$.
(2) $\bar{2}{\cdot}9456 \times 3$. (5) $\bar{2}{\cdot}9947 \times 0{\cdot}8$.
(3) $\bar{1}{\cdot}5782 \times 5$. (6) $\bar{2}{\cdot}7165 \times 2{\cdot}5$.

7. Find the values of:

(1) $\bar{3}{\cdot}9778 \times 0{\cdot}65$. (4) $2{\cdot}1342 \times -0{\cdot}4$.
(2) $\bar{2}{\cdot}8947 \times 0{\cdot}84$. (5) $1{\cdot}3164 \times -1{\cdot}5$.
(3) $\bar{1}{\cdot}6257 \times 0{\cdot}6$. (6) $\bar{1}{\cdot}2976 \times -0{\cdot}8$.

8. Find the values of:

(1) $\bar{1}{\cdot}4798 \div 2$. (4) $\bar{3}{\cdot}1195 \div 2$.
(2) $\bar{2}{\cdot}5637 \div 5$. (5) $\bar{1}{\cdot}6173 \div 1{\cdot}4$.
(3) $\bar{4}{\cdot}3178 \div 3$. (6) $\bar{2}{\cdot}3178 \div 0{\cdot}8$.

Section F. Logarithmic Calculations

Use logarithms to find the values of the following:

1. $15{\cdot}62 \times 0{\cdot}987$.
2. $0{\cdot}4732 \times 0{\cdot}694$.
3. $0{\cdot}513 \times 0{\cdot}0298$.
4. $75{\cdot}94 \times 0{\cdot}0916 \times 0{\cdot}8194$.
5. $9{\cdot}463 \div 15{\cdot}47$.
6. $0{\cdot}9635 \div 29{\cdot}74$.
7. $27{\cdot}91 \div 569{\cdot}4$.
8. $0{\cdot}0917 \div 0{\cdot}5732$.
9. $5{\cdot}672 \times 14{\cdot}83 \div 0{\cdot}9873$.
10. $(0{\cdot}9173)^2$.
11. $(0{\cdot}4967)^3$.
12. $\sqrt[3]{1{\cdot}7135}$.
13. $\sqrt[4]{647{\cdot}2} \div (3{\cdot}715)^3$.
14. $\frac{1}{2}(48{\cdot}62)^{\frac{1}{3}}$.
15. $\sqrt[3]{\dfrac{9{\cdot}728}{3{\cdot}142}}$.
16. $(1{\cdot}697)^{2{\cdot}4}$.
17. $(19{\cdot}72)^{0{\cdot}57}$.
18. $(0{\cdot}478)^{3{\cdot}1}$.
19. $(5{\cdot}684)^{-1{\cdot}12}$.
20. $(0{\cdot}5173)^{-3{\cdot}4}$.

21. $\dfrac{3{\cdot}142 \times 6{\cdot}97^2 \times 1500}{473{\cdot}8}$.

22. $\sqrt{\dfrac{69{\cdot}8 \times 0{\cdot}0579 \times 53{\cdot}2}{1476}}$.

23. $\dfrac{51{\cdot}72 \times (8{\cdot}63)^3}{964{\cdot}8}$.

24. $\sqrt{(5{\cdot}673)^2 + (9{\cdot}28)^2}$.

25. $\sqrt{15{\cdot}78^2 - (14{\cdot}17)^2}$.

SECTION G. MISCELLANEOUS

1. (*a*) Without the use of logarithm tables or slide rules evaluate:

$$\text{(i) } 27^{\frac{2}{3}}; \quad \text{(ii) } \tfrac{1}{16}^{-\frac{3}{4}}; \quad \text{(iii) } \frac{16 \times 10^{-2}}{\sqrt{\mathrm{LC}}},$$

where $\mathrm{L} = 3{\cdot}125 \times 10^{-6}$ and $\mathrm{C} = 200 \times 10^{-12}$.

(*b*) Using logarithm tables evaluate:

$$\text{(i) } \sqrt{23{\cdot}2^2 + 16{\cdot}8^2}; \quad \text{(ii) } \sqrt[5]{0{\cdot}0863}.$$

(Coventry.)

2. Evaluate the following, using logarithms. Give your results correct to three significant figures:

(*a*) $\sqrt[4]{\dfrac{1}{0{\cdot}873}}$. (*b*) $(1{\cdot}3)^{1{\cdot}8}$.

(*c*) $\dfrac{6{\cdot}172 \times 0{\cdot}1941}{(0{\cdot}9835)^2}$. (*d*) $(2{\cdot}73)^0$.

(Handsworth.)

3. (i) Use logarithms to evaluate

$$\sqrt[4]{\left(\frac{24{\cdot}36 + 8{\cdot}07}{24{\cdot}36 - 8{\cdot}07}\right)^3}.$$

(ii) The load-carrying capacity of a gear tooth is given by the formula

$$\mathrm{W} = \frac{600\mathrm{S}pf\mathrm{Y}}{600 + \mathrm{V}}.$$

Calculate W when $\mathrm{S} = 7500$, $p = 0{\cdot}5$, $f = 1{\cdot}75$, $\mathrm{Y} = 0{\cdot}105$, $\mathrm{V} = 500$. (Sunderland.)

4. (*a*) Evaluate, using logarithms,

(i) $\dfrac{314{\cdot}2 \times 0{\cdot}00684}{0{\cdot}01098}$;

(ii) $\sqrt{(20{\cdot}335)^2 - (5{\cdot}505)^2}$.

(*b*) Find x if $\log_{10}(2x - 1) = 1$. (U.L.C.I.)

5. Use tables to evaluate correct to three significant figures:

(*a*) $\sqrt{(14{\cdot}32)^2 + (13{\cdot}27)^2}$;

(*b*) $\dfrac{1}{18{\cdot}63} + \dfrac{3}{21{\cdot}45}$;

(*c*) $\dfrac{1{\cdot}729 \times (21{\cdot}62)^3}{517{\cdot}4}$;

(*d*) $\dfrac{\sqrt{26{\cdot}82} + \sqrt{147{\cdot}3}}{\sqrt{268{\cdot}2} - \sqrt{14{\cdot}73}}$.

(E.M.E.U.)

6. Simplify: (i) $a^m \times a^n$;
(ii) $a^m \div a^n$;
(iii) $(a^m)^n$;
(iv) How is a meaning given to a^0 using the answers given? (W.R. Yorks.)

7. (*a*) Express each of the numbers 2, 50, 100, 50^3 as a power of 10.

(*b*) Evaluate $100^{1{\cdot}24}$, $\sqrt[5]{100}$.

(*c*) By means of logarithms evaluate

$$\frac{\sqrt{467{\cdot}2} \times \sqrt[3]{7{\cdot}3}}{\sqrt[3]{467{\cdot}2} \times \sqrt{7{\cdot}3}}$$

(N.C.T.E.C.)

8. Without using tables find the values of:

(*a*) $\log 27 \div \log 3$. (*b*) $(\log 16 - \log 2) \div \log 2$.

(U.L.C.I.)

9. Find the values of

$$10^{0{\cdot}544}, \log (63 - 21), \log 63 - \log 21,$$

and express 5 as a power of 10. (U.E.I.)

10. (1) Express each of the numbers 1·75, 175, $6{\cdot}73^2$ as a power of 10.

(2) Evaluate $200^{0{\cdot}8}$.

(3) Evaluate $\dfrac{78{\cdot}3\sqrt[3]{5{\cdot}73}}{\sqrt{7{\cdot}83}}$. (N.C.T.E.C.)

11. Find the value of $\frac{4}{3}\pi r^3$ when $\pi = 3{\cdot}142$ and $r = 2{\cdot}90$. (U.E.I.)

12. Find the value of $\sqrt{(9{\cdot}485)^2 - (5{\cdot}475)^2}$.

13. In the formula $V = \sqrt{\dfrac{2ghD}{0{\cdot}03L}}$, find V when $g = 32{\cdot}2$, $h = 0{\cdot}627$, $L = 175$, $D = 0{\cdot}27$.

14. If $V^{1{\cdot}0646} = \dfrac{479}{P}$, calculate V when $P = 30$. (U.E.I.)

15. Find the value of y when

$$y = \frac{23{\cdot}31}{4} + \sqrt{(5{\cdot}708)^2 \div (3{\cdot}393 \times 27{\cdot}18)}$$

(U.L.C.I.)

16. The area of a triangle is given by the formula area $= \sqrt{s(s-a)(s-b)(s-c)}$ where a, b and c are sides of the triangle and $s = \frac{1}{2}(a + b + c)$.

Find the area of the triangle when $a = 30{\cdot}65$ in., $b = 51{\cdot}98$ in., $c = 25{\cdot}46$ in.

(Students may be interested to note that considerable manipulation of expressions bringing in the lengths of the sides, and the trigonometrical functions of the angles (see Chapter 10) has been necessary to lead to this formula for the area of a triangle. The form is used because the four factors under the root sign are easily calculated, and once these are known the multiplication of the factors, and the extraction of the root, are operations readily carried out with the aid of logarithms. See Example 5 on p. 182.)

17. If $V = pv^{1{\cdot}6}$ find V when $v = 6{\cdot}032$, $p = 29{\cdot}12$.

18. If $R^n = 1{\cdot}8575$, find R when $n = 18$.

19. By means of logarithms or otherwise, find the value of:

(a) $\dfrac{9{\cdot}32 \times 0{\cdot}761}{\sqrt{18{\cdot}2}}$. (c) $(1{\cdot}34)^{1{\cdot}2}$.

(b) $(18{\cdot}56)^{\frac{1}{5}}$. (d) $(19{\cdot}75)^2 - (16{\cdot}75)^2$.

(U.E.I.)

20. (a) Find (using logs) $0{\cdot}0387^{-1{\cdot}4}$.

(b) If $T = \dfrac{\pi^2 n^2 a^2 \omega}{900g}$ express n in terms of π, a, ω, g and T and calculate the value of n when $\omega = 7{\cdot}75$, $a = 1{\cdot}33$, $g = 32{\cdot}2$ and $T = 187$.

(E.M.E.U.)

CHAPTER 9

FUNDAMENTAL GEOMETRIC TRUTHS

SECTION A

1. General Idea of an Angle

What is an angle? Looked at from the simplest point of view we can take it as being formed by the intersection of two lines. But there is more than this to be considered, because of the part that angles play in Geometry and Trigonometry. Our ideas must be more precise, and more in accordance with the requirements of these two branches of Mathematics.

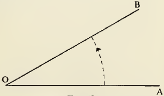


FIG. 41.

Let AOB (Fig. 41) represent an angle. We can imagine this angle as being formed by a line which starts from the position OA, and rotates about O in a contra-clockwise direction to the position OB.

In such a case O is the **vertex** of the angle, while OB and OA are called the **arms** of the angle.

Assuming the rotation to be **contra-clockwise,** we regard the angle so formed as being **positive**; if **clockwise,** we regard it as a **negative** angle.

With any given angle we generally assume the rotation to be in a **contra-clockwise** direction unless otherwise stated.

Let us assume the line OA to make a complete revolution in four equal stages in the direction shown in Fig. 42.

We then have four equal angles, each of which is called a *right angle*.

Then we have

$\angle AOB_1 = 1$ right angle
$\angle AOB_2 = 2$ right angles
$\angle AOB_3 = 3$ right angles

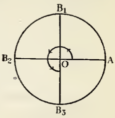


FIG. 42.

2. Measurement of Angles

The *right angle*, though it provides us with a kind of natural unit for measuring angles, is too large a unit for general use.

We therefore assume the rotation to take place in 360 equal steps called degrees, so that **one right angle** $= 90°$.

The degree is subdivided into 60 minutes, or 60′. The minute is subdivided into 60 seconds, or 60″, so that

60 seconds = 1 minute,
60 minutes = 1 degree,
90 degrees = 1 right angle.

In the Metric system the right angle is divided into 100 equal parts.

3. Acute, Obtuse, and Reflex, Angles

An **Acute** angle is one which is less than a right angle.

An **Obtuse** angle is one which is greater than a right angle but less than two right angles.

A **Reflex** angle is an angle which is greater than two right angles.

4. Complementary and Supplementary Angles

(*a*) If two angles together make up **one** right angle or 90°, they are said to be **complementary** angles, and each angle is called the **complement** of the other.

(*b*) On the other hand, if the sum of two angles be **two** right angles or 180°, they are said to be **supplementary,** each angle being the **supplement** of the other.

Hence 17° and 73° are **complementary** angles and 108° and 72° are **supplementary** angles.

5. The Angles between a Straight Line and a Plane

Let CDEF (Fig. 43) represent one surface of a drawing-board, which for our purpose we may consider as being part

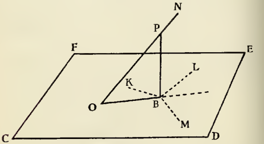


FIG. 43.

of a **horizontal** plane, and let ON be a very thin rod held obliquely but meeting the plane at O.

Let any point P be taken in ON and let a **vertical** line be drawn through it, *i.e.* a line which is also **perpendicular** to the plane.

Let this line meet the plane in B. Join OB. Then $\angle$POB is called the **angle between the line ON and the plane.**

Also OB is called the **projection** of OP upon the plane.

The angle between a straight line and a plane is the angle between the line itself and its projection on the plane.

Further, PB is perpendicular to any line such as BL, BK or BM passing through B, provided those lines lie wholly within the plane.

NOTE.—*A straight line is said to be perpendicular to a plane when it is perpendicular to any straight line which it meets in the plane.*

Let ABC (Fig. 44) represent the end elevation of the top of a desk where AB is at right angles to CB.

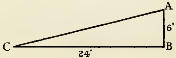


FIG. 44.

Then the $\angle$ACB is the angle between a horizontal plane which we can denote by CB, and a plane which we can denote by CA.

Such an angle is called **the slope of the line CA** or of the corresponding plane.

Now, the ratio $\frac{AB}{BC}$ represents what is called **the gradient** of the line CA.

Fig. 44 shows a rise of 6 in. for a horizontal distance of 24 in., so that the **gradient** in this case is $\frac{6}{24}$ or $\frac{1}{4}$.

In other words, we can say that the line CA rises 1 in 4.

In certain cases where the angle is very small, and where at the same time CA will differ but slightly from CB, the gradient is sometimes taken as $\frac{AB}{AC}$.

This is how a motorist or a railway engineer considers a gradient.

When the former speaks of a road having a gradient of 1 in 10, he means that for every 10 ft along the road there is a rise of 1 ft.

On a railway also, one may see an indicator which shows that the line, in one part, rises 1 in 200.

The ratio $\frac{1}{200}$ is considered by the engineer to be the gradient of the line in that particular part.

7. Angles Formed by Two Intersecting Straight Lines

Two theorems which deal with two straight lines which meet or intersect are submitted without proof.

(1) *When one straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.*

(2) *When two straight lines intersect, the vertically opposite angles are equal.*

8. Parallel Straight Lines

So far, in our treatment of angles, we have dealt with lines which have undergone a certain amount of rotation, and however slight that rotation may have been, the initial line in its new position has been changed in direction.

Let us now consider the case of a line, moving, but without rotation, *i.e.* without change of direction.

An example of this is provided by sliding a set-square along a fixed straight edge of a rule (see Fig. 45).

PQR and $P_1Q_1R_1$ represent two positions of the set-square when this has been done. Then the edge PQ of the set-square represents a straight line moving into a new position P_1Q_1, but without rotation and in a direction parallel to itself.

PQ and P_1Q_1 are then said to be parallel straight lines.

Again, as the set-square slides along, it is obvious that the inclination of PQ to the edge AB of the ruler is constant so that the $\angle PQB = \angle P_1Q_1B$.

These angles are known as **corresponding angles.**

The line AB which crosses the parallel lines PQ and P_1Q_1 is called a **transversal.**

We see, then, *that two straight lines lying in the same plane are parallel if a transversal to them makes the corresponding angles equal.*

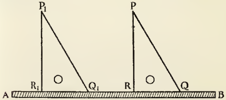


FIG. 45.

Parallel Straight Lines may be defined as straight lines which are in the same plane, and do not meet, however far they may be produced in either direction.

9. Fundamental Properties of Parallel Straight Lines

Let AB and CD (Fig. 46) be two parallel straight lines, and let EH be any transversal drawn across them. For the sake of brevity the various angles are denoted by numerals as shown.

An examination of this figure shows:

(1) $\angle 2 = \angle 4$, $\angle 6 = \angle 8$, $\angle 1 = \angle 3$, $\angle 7 = \angle 5$.

Such pairs of angles are called **corresponding** angles.

(2) $\angle 6 = \angle 2$, $\angle 3 = \angle 7$.

These pairs are termed **alternate** angles.

(3) $\angle 3 + \angle 2 = 2$ right angles.
$\angle 6 + \angle 7 = 2$ right angles.

Summarised we can state these observations as follows:

When a **transversal** intersects two parallel straight lines

(1) *The corresponding angles are equal.*
(2) *The alternate angles are equal.*
(3) *The sum of the interior angles on the same side of the transversal is equal to two right angles.*

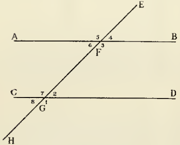


FIG. 46.

Conversely, if a transversal crosses two straight lines and any one of these three conditions holds good, the two straight lines must be parallel.

SECTION B. RECTILINEAL FIGURES

10. Rectilineal Figures are Plane Figures which are Bounded by Straight Lines

As the student is assumed to have a working knowledge of simple rectilineal figures, and many have already been dealt with in earlier chapters, special reference to any particular type or class will be omitted.

11. Symmetry—Symmetrical Figures

A plane figure is said to be **symmetrical** about a straight line, if on folding the figure about that line the two parts on either side of the line can be made to coincide.

The straight line is called the **axis of symmetry.**

Examples.

1. A square is symmetrical about a diagonal, and also about a line which bisects two opposite sides at right angles.
2. A circle is symmetrical about a diameter.
3. A regular hexagon is a symmetrical figure which has six axes of symmetry. The reader should trace these.

12. Angle Properties of a Triangle

The following statements are submitted without proof.

(1) *When a side of a triangle is produced the exterior angle thus formed is equal to the sum of the two interior opposite angles.*

(2) *The sum of the angles of a triangle is equal to two right angles or* $180°$.

The student should note that these theorems are of great importance.

It follows quite simply from the above that the four angles of any quadrilateral are together equal to four right angles, since if one diagonal be drawn two triangles will be obtained and the sum of the angles of each triangle is two right angles.

13. Theorems Relating to Congruency of Triangles

Two triangles are congruent if one can be superimposed on the other, so that they exactly coincide with regard to their vertices or angular points, and their sides.

Their areas must consequently be equal. In other words,

the three sides of one triangle must have the same lengths as the three sides of the other, each to each, and the angles of the triangles opposite to the equal sides must also be equal.

Case I

Two triangles are congruent if the three sides of the one are respectively equal to the three corresponding sides of the other.

Case II

Two triangles are congruent if they have two sides equal each to each, and if the included angle of the one is equal to the included angle of the other.

Case III

Two triangles are congruent if they have two angles equal each to each, and a corresponding side in each triangle equal.

14. Two Theorems Relating to Triangles which are Self-Evident

I. Any two sides of a triangle are together greater than the third side.

II. The greater side of a triangle is opposite to the greater angle.

15. The Theorem of Pythagoras

This theorem can be stated as follows:

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

NOTE.—The hypotenuse is the side of the triangle opposite the right angle.

As the theoretical proof for this theorem is rather involved, we do not propose to give it here, but will illustrate

the theorem by a method which is based on facts in Mensuration and Algebra already dealt with (see Fig. 47).

ABCD is a square (Fig. 47).

Equal distances CE, DH, AG and BF are marked off from the angular points, each being equal to a.

Let b = the length of the remaining portion of each side.

If E, F, G and H be joined another square is obtained.

Let c represent the length of one side of this square.

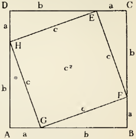


FIG. 47.

$$\text{Area of square ABCD} = (a + b)^2 = a^2 + 2ab + b^2$$

$$\text{Area of each triangle} = \tfrac{1}{2}ab$$

$$\text{Area of four triangles} = 4 \times \tfrac{1}{2}ab = 2ab$$

$$\text{Area of inner square} = c^2$$

Now area of inner square

$$= a^2 + 2ab + b^2 - \text{area of triangles}$$

$$= a^2 + 2ab + b^2 - 2ab$$

$$\therefore \quad c^2 = a^2 + b^2$$

Consider the $\triangle$ECF.

Now, a^2 = area of square on side EC.

b^2 = area of square on side CF.

and c^2 = area of square on side EF.

Hence the square on the hypotenuse EF is equal to the sum of the squares on EC and CF.

This theorem can be illustrated experimentally, and one of the methods, which may be of interest to the student, is set out, and explained below.

A right-angled triangle such as ABC (Fig. 48) with AC as hypotenuse, is drawn on drawing paper or card-board.

Squares are then constructed on its sides as shown.

The centre O of the square on BC is found, and through O, LM is drawn parallel to AC, and PQ perpendicular to AC.

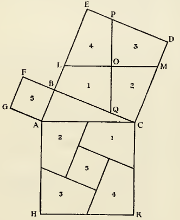


FIG. 48.

This construction gives four quadrilaterals, which are numbered 1, 2, 3 and 4.

These quadrilaterals and the square on AB are cut out from the paper, and are superimposed as shown on the square which has been constructed on the hypotenuse AC. It will be found that the five figures will exactly fit into the square on the hypotenuse AC, and therefore have the same area as that square.

This theorem is of considerable importance, and can be

employed to find one side of a right-angled triangle when the other two sides are known.

Example 1. *Find the area of an equilateral triangle in terms of its side.*

Let ABC be an equilateral triangle with AD drawn perpendicular to the base BC (Fig. 49).

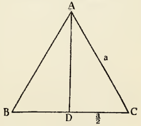


FIG. 49.

Let each side of the triangle $= a$.

Then $\quad BD = DC = \frac{a}{2}$.

Since $\triangle ADC$ is right-angled at D

$$AD^2 = AC^2 - DC^2$$

that is
$$AD^2 = a^2 - \left(\frac{a}{2}\right)^2$$
$$= a^2 - \frac{a^2}{4}$$
$$= \frac{3a^2}{4}.$$

Hence
$$AD = \frac{\sqrt{3}}{2}a.$$

Now, area of the equilateral triangle $= \frac{1}{2}BC \times AD$

$$= \frac{a}{2} \times \frac{\sqrt{3}a}{2}$$

$$= \frac{\sqrt{3}a^2}{4}.$$

Example 2. *A ladder 40 ft long rests against a house so that the foot of the ladder is 14 ft from the foot of the wall. How far up does it reach?*

Let AB represent the ladder and AC the wall (Fig. 50).

Now $c^2 = a^2 + b^2$

$\therefore \quad 40^2 = 14^2 + b^2$

that is $b^2 = 40^2 - 14^2$

$b^2 = 1404$

$\therefore \quad b = 37{\cdot}5$ ft approx.

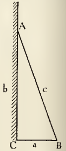


Fig. 50

16. Similar Triangles

If two triangles have the **three angles** of the one respectively equal to the **three angles** of the other they are not necessarily congruent.

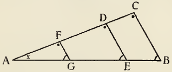


Fig. 51

Such triangles are said to be **Similar.**

Let ABC be a triangle with points G and E in base AB (Fig. 51).

Draw GF and ED parallel to BC.

Then $\angle AFG = \angle ADE = \angle ACB$ (corresponding angles).
$\angle AGF = \angle AED = \angle ABC$ (corresponding angles).

The angle at A is common to the three triangles AFG, ADE and ACB.

If we consider any pair of these triangles, we notice that the three angles of the one are respectively equal to the three angles of the other, but obviously they are **not congruent.**

All three triangles are said to be **similar** because they are **equiangular.**

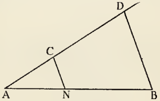


Fig. 52.

In more advanced geometry it is shown that when two triangles are equiangular the ratios of **corresponding sides** are equal.

Applying this theorem to Fig. 51 we can say that

$$\frac{AF}{AG} = \frac{AD}{AE} = \frac{AC}{AB}$$

So also it follows that $\frac{AF}{AD} = \frac{AG}{AE}$

and $\frac{AG}{AE} = \frac{FG}{DE}.$

Note carefully that by corresponding sides we mean those sides that are opposite equal angles.

For **rectilineal figures** in general we can now say that **Similar figures are figures in which the ratios of corresponding sides or lengths are equal and corresponding angles are equal.**

Example. *Divide a given line in the ratio of* $2:3$ *or* $\frac{2}{3}$.

Let AB be the given line.
Draw any line AE making $\angle EAB$ with AB (Fig. 49).
Mark off along AE a distance AC = 2 units of length and CD = 3 units. Join DB, and draw CN parallel to DB.

Now $$\frac{AC}{CD} = \frac{2}{3}$$

$$\therefore \quad \frac{AC}{AD} = \frac{2}{5}$$

Because CN and DB are parallel, the $\triangle$s ACN and ADB are equiangular, and therefore similar.

Hence $$\frac{AN}{AB} = \frac{AC}{AD} = \frac{2}{5}$$

$$\therefore \quad \frac{AN}{NB} = \frac{2}{3}$$

that is N is the required point.

NOTE.—The principle involved in this and similar examples is made use of in the construction of scales, and more particularly in the case of the diagonal scale.

Arising out of the theorem enunciated and explained above, it can be shown that **the ratio of the heights of two similar triangles is the same as the ratio of any pair of corresponding sides.**

Algebraically if h and h_1 represent the heights, and b and b_1 represent the bases, the above becomes

$$\frac{h}{h_1} = \frac{b}{b_1}$$

17. Relation between the Areas of Similar Triangles

Let $A_1B_1C_1$ and $A_2B_2C_2$ be similar triangles (Fig. 53).
Let h_1 and h_2 be their respective heights and a_1 and a_2 their bases.

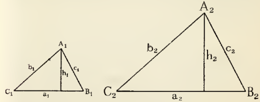


FIG. 53.

Since the triangles are similar

$$\frac{h_1}{a_1} = \frac{h_2}{a_2}$$

that is $$h_1 = \frac{a_1}{a_2} \cdot h_2 \quad \ldots \ldots \quad (1)$$

Also, $$\frac{\text{Area of } \triangle A_1B_1C_1}{\text{Area of } \triangle A_2B_2C_2} = \frac{\frac{1}{2}a_1h_1}{\frac{1}{2}a_2h_2} \quad \ldots \quad (2)$$

Substituting for h_1 as shown in (1), we have:

$$\frac{\text{Area of } \triangle A_1B_1C_1}{\text{Area of } \triangle A_2B_2C_2} = \frac{\frac{1}{2}a_1 \times \frac{a_1}{a_2} \cdot h_2}{\frac{1}{2}a_2h_2}$$
$$= \frac{a_1^2}{a_2^2}$$

Hence the areas of the triangles are proportional to the squares of the corresponding sides C_1B_1 and C_2B_2.

Similarly the areas could be proved proportional to the squares of B_1A_1 and B_2A_2.

It follows from this that the areas are also *proportional to the squares of the corresponding heights.*

Furthermore it can be shown by more advanced geometry that **the areas of all similar rectilineal figures are proportional to the squares of their corresponding sides.**

SECTION C. IMPORTANT GEOMETRICAL TRUTHS RELATING TO CIRCLES

18. It is desirable at this stage to refer to certain terms and definitions relating to the Geometry of the Circle.

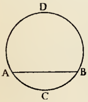


FIG. 54.

(1) A **Chord** of a circle is any straight line which divides the circle into two parts, and is terminated at each end by the circumference.

(2) An **Arc** of a circle is a portion of the circumference.

(3) A **Segment** of a circle is a figure bounded by a chord and the arc which it cuts off.

In Fig. 54 the chord AB divides the circle into two segments.

(i) The **minor** segment ACB.
(ii) The **major** segment ADB.

(4) A **Sector** of a circle is a figure which is bounded by two radii and the arc between them.

(5) The **Angle in a segment** is the angle subtended at a point on the arc of the segment by the chord of the segment.

For example, the angle ADB (Fig. 55) is the angle in the **major** segment, while the angle ACB is the angle in the **minor** segment.

Each angle is subtended by the chord AB.

(6) By the **angle at the centre** we mean the angle subtended at the centre by a chord or by an arc.

In Fig. 55 the $\angle AOB$ is the angle subtended at the centre by the chord AB and by the arc ACB.

19. The following theorem is submitted but without proof.

The angle at the centre of a circle subtended by an arc is double the angle at the circumference subtended by the same arc.

Thus in Fig. 55 the $\angle AOB = 2\angle ADB$
and the reflex $\angle AOB = 2\angle ACB$.

Some very important results follow from this Theorem.

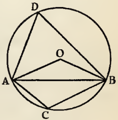


FIG. 55.

I. All angles in the same segment of a circle are equal.

II. The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles; that is, they are supplementary.

III. The angle in a semi-circle is a right angle.

IV. In equal circles, arcs which subtend equal angles either at the centres or at the circumferences are equal.

V. In equal circles, chords which cut off equal arcs are equal.

20. Tangents to a Circle

(1) A tangent to a circle is a straight line which meets the circle in one point called the point of contact and does not cut the circle when produced.

(2) A tangent to a circle is at right angles to the radius drawn from the point of contact.

In Fig. 56, QR is a tangent at right angles to the radius OP at the point of contact P.

21. The following Theorems are substituted without proofs.

I. The angle between a tangent and a chord drawn through the point of contact is equal to the angle in the alternate segment.

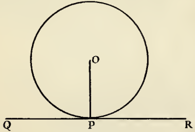


Fig. 56.

In Fig. 57 AC is the tangent
BD is the chord.

Then $\angle DBC = \angle DEB$
and $\angle ABD = \angle DFB$

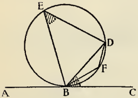


Fig. 57.

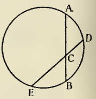


Fig. 58.

II. If two chords intersect in a circle the product of the segments of one chord is equal to the product of the segments of the other.

Thus in Fig. 58,

$$AC \times CB = DC \times CE.$$

Example. *The diameter of a circle is 4 in. It bisects a chord at right angles so that the height of one segment is 1·2 in. Find the length of the chord.*

In Fig. 59 AB is a diameter and DE a chord.

BC = 1·2 in. Then AC = 2·8 in.

From the theorem above, § **21, II**

$$DC \times CE = AC \times CB$$

or $$DC^2 = 1{\cdot}2 \times 2{\cdot}8$$

that is $$DC^2 = 3{\cdot}36$$

$$\therefore\ DC = 1{\cdot}83$$

Hence length of chord DE = 3·66 in.

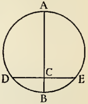


Fig. 59.

III.

In Fig. 60 PA is a tangent to the circle, and PDB and PEC are any two lines drawn from P cutting the circle at D and E respectively.

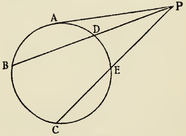


Fig. 60.

It can be shown that

$$PA^2 = PD \times PB = PE \times PC.$$

EXERCISE IX

Angles

1. A clock is started at noon. Through what angles will the minute hand have turned by: (1) 2.45 p.m.? (2) 10 minutes past 4? At what time will it have turned through 192°?

2. A patternmaker carves a straight groove of semi-circular section across the flat face of a piece of timber. How can he check his accuracy by means of a set-square?

3. $x°$ is the smallest angle in a triangle. If the others are $2{\cdot}5x°$ and $4{\cdot}5x°$, what is the value of each angle?

4. The angles A, B, C and D of a four-sided figure are 135°, 110°, 70° and 45°. If BE be drawn parallel to AD meeting DC at E, find the angles at B and E of the new figure. What is the original figure?

Theorem of Pythagoras

5. A man travels 15 miles due east and then 18 miles due north. How far is he from his starting point?

6. A builder has a 2-ft rule and a supply of laths (strips of wood). How can he easily construct a square which he could use to test the accuracy of corners in his building?

You probably know the answer to this question. It was the method used by the builders of the pyramids.

7. The diameter AB of a semi-circle is 2·5 in. in length. A point P on the circle is 1·8 in. from A. How far is P from B?

8. A chord of a circle is 2·6 in. long and the distance of the centre from it is 1·1 in. Find the radius of the circle.

9. A fitter prepares four bars, of lengths 3 ft, 2 ft, 3 ft and 2 ft respectively, which he has to rivet together to make a rectangular frame. How can he check that he has joined them truly at right angles?

10. A point P is 5 in. from the centre of a circle of radius 2·5 in. What is the length of the tangent from P?

11. Show that if triangles have their sides in the ratio of (1) 3 : 4 : 5, (2) 5 : 12 : 13, the triangles are right-angled.

12. A peg is 15 ft from the foot of a flagstaff which is 40 ft high. What length of rope will be needed to stretch from top of the flagstaff to the peg?

13. Find the length of the diagonal of a square field whose area is 10 acres.

Similar Figures

14. The adjacent sides of a rectangle are 3 in. and 4 in. respectively. Find the diagonal of a similar rectangle whose longest side is 17·8 in.

15. In a triangle ABC, AB = 3·5 in., BC = 2·4 in. and AC = 3·2 in. DE is drawn parallel to BC, so that AD = 2·8 in. Find the other sides of the triangle ADE.

16. ABC is a right-angled triangle, with its right angle at B, and its sides AB and BC are in the ratio of 2 : 1. If AB is produced to F so that AF = 200 yd and FE is drawn perpendicular to AC produced, what are the lengths of FE and AE.

17. Three 60°–30° set-squares are made from suitable close-grained easily-worked thin wood. How could you adjust the 3 angles to accuracy without the use of a protractor?

18. The triangle ABC is right-angled at B. If AB = 17 in. and BC = 9 in., find BD the length of the perpendicular to the hypotenuse, and also the lengths of AD and DC.

19. The area of a triangle is 17·84 sq in. and its height is 5·6 in. Find the area of a similar triangle whose corresponding height is 7·8 in.

20. If on a map $\frac{1}{4}$ sq in. represents 4 sq miles,

(a) To what scale is the map drawn?
(b) What would be the distance on the map of two points $15\frac{3}{4}$ miles apart?

21. BCD is a right-angled triangle with its right angle at C. BC = 0·9 in. and CD = 1·3 in. BD is produced to F so that $\frac{BD}{BF} = \frac{3}{7}$, and BF is the hypotenuse of a similar triangle.

Find (1) the length of BF.
(2) The area of the larger triangle.

Miscellaneous

22. An iron plate has the form of an equilateral triangle, each side of which is 10 in. long. Calculate the radius of the largest circle that can be cut out of the plate. (Handsworth.)

23. The angles of a triangle are x, $2x$ and $(4x - 30)$ degrees respectively. Find the angles. (Handsworth.)

24. AB is the vertical diameter of a circle whose centre is O. With centre A and any radius an arc is drawn cutting the circle in two points C and D. CD, OC, OD, AC and AD are joined.

Prove (*a*) the triangles AOC and AOD are congruent, (*b*) angle BOC = twice angle BAC. (U.L.C.I.)

25. AD is drawn tangential to a circle at A, and two chords AC and AB are drawn such that angle DAC is 65° and angle DAB is 108°. If the points B and C are joined, determine angle BCA. If O is the centre of the circle, determine angle AOB. (U.L.C.I.)

26. Find α in the compound V-block shown in Fig. 61. (Coventry.)

27. A circle has a 4 in. radius; calculate (i) its circumference, (ii) the length of arc subtended by an angle of 60° at the centre. (iii) A sector of a circle 2 in. diameter is equal in area to that of a full circle 1 in. diameter. What is the angle contained by the sector of the larger circle? (W.R. Yorks.)

28. Answer the following questions covering the geometrical properties of a circle:

(i) What relationship exists between the angles in the same segment?
(ii) What relationship exists between intersecting chords?

Make sketches to show you understand these questions. (W.R. Yorks.)

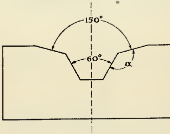


FIG. 61.

29. Fig. 62 represents a rectangular room, whose dimensions are:

AB = 12·3 ft
BC = 9·4 ft
BF = 9·0 ft

Find the distance from B to H:

(i) diagonally across the ceiling and down DH;
(ii) direct. (S.W. Essex.)

30. If the height of the arch of a bridge is h and the span is $2s$, show that $2rh - h^2 = s^2$, if r be the radius of the arch. Also find r if $h = 12$ ft and $s = 36$ ft.

31. The chord of a circle is 10 in. long, and the radius

is 6 in. long. Find the height of each arc into which the circle is divided.

32. Two straight lines MN and KL intersect at O. MO = 2·6 in., ON = 1·1 in. and OK = 1·8 in. If M, K, N and L lie on a circle find the length of OL.

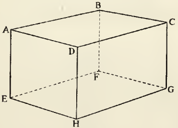


FIG 62

33. An octagonal room has an area of 440 sq ft. If its plan is drawn to a scale so that 1 in. represents 5 ft, what will be the area of the plan?

If the area of the plan is increased in the ratio of 9 to 4, what is the new scale?

34. In a circle 4·5 in. diameter place a chord 3·5 in. long. Draw a diameter at right angles to the chord. Measure the segments of the chord and of the diameter. Find the product of the segments of each and compare the results. (U.L.C.I.)

35. The lengths of the sides of a rectangle are x in. and y in. If p in. and d in. denote its perimeter and the length of its diagonal respectively, express p^2 and $4d^2$ in terms of x and y and show that $\frac{1}{8}(p - 2d)(p + 2d)$ sq in. equals the area of the rectangle. Calculate the area of a rectangle whose perimeter and diagonal equal 46 in. and 17 in. respectively. (N.C.T.E.C.)

CHAPTER 10

TRIGONOMETRICAL RATIOS

1. Experimental Determination of a Height

Let AB, Fig. 63, represent the height h of a factory chimney. The value of h can be determined experimentally as follows:

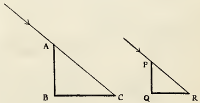


FIG. 63.

Let BC represent the shadow of AB, the direction of the sun's rays being shown by AC.

Let PQ, a rod of known length, stand vertically and let QR be its shadow. The angle PRQ is called the altitude of the sun.

The sun's rays being parallel, PR will be parallel to AC

$$\therefore \quad \angle PQR = \angle ACB$$

$\therefore$ $\triangle$s ABC, PQR are similar (see p. 208)

$$\therefore \quad \frac{AB}{BC} = \frac{PQ}{QR} \text{ and } AB = BC \times \frac{PQ}{QR}$$

BC, PQ and QR can be measured and so AB may be found.

Let BC = 105 ft, PQ = 3 ft, QR = 3·5 ft.

Then substituting $\quad \text{AB} = 105 \times \frac{3}{3\cdot5}$

whence $\quad \text{AB} = 90 \text{ ft.}$

It is clear that whatever be the length of PQ, we shall obtain the same answer, because

the ratio $\frac{\text{PQ}}{\text{QR}}$ *will be constant for the angle* R *or* C.

If we knew the value of this ratio, the problem could be solved without the use of PQ.

We need only measure $\angle$ACB *and the length* BC.

2. The Tangent of an Angle

Let a straight line OA rotate in an anti-clockwise direction from a fixed line OX (Fig. 64).

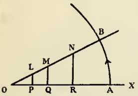


Fig. 64

Let OB be any position taken up by the rotating line and let $\angle$AOB be the angle so formed.

Take any points L, M, N on OB and draw LP, MQ, NR perpendicular to OX.

Then, as shown in Chapter 9,

$$\triangle\text{s LOP, MOQ, NOR are similar}$$

$$\therefore \quad \text{ratios } \frac{\text{LP}}{\text{OP}}, \frac{\text{MQ}}{\text{OQ}}, \frac{\text{NR}}{\text{OR}} \text{ are equal.}$$

However many points are taken on OB, the value of this ratio for the angle AOB will be the same.

A similar result can be obtained for any other angle.

$\therefore$ *Each angle has its own constant ratio by which it can be identified.*

This constant ratio is called the tangent of the angle.

NOTE. *Notation for angles.* So far angles have been denoted by the use of three letters. For brevity, only the letter at the vertex is often employed. When dealing generally with angles, instead of letters, we refer to an angle by using special symbols. Very commonly we employ the Greek letters θ (theta), α (alpha), etc. Such letters are used in much the same way as letters such as x, y, k, etc., in algebra.

3. Right-Angled Triangles

In Fig. 64 several right-angled triangles were constructed, in each of which a ratio was obtained which represented the tangent of the angle at O. We will now consider more formally the relations which exist between the sides and angles of any right-angled triangle.

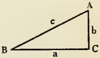


Fig. 65.

In Fig. 65 let ABC be a right-angled triangle. Let the sides opposite to the angles A, B and C be connoted by the small letters a, b, c.

Then, as shown in § 2,

$$\tan \text{ABC} = \frac{\text{side opposite}}{\text{side adjacent}}$$

or

$$\tan \text{ABC} = \frac{b}{a}$$

$$\therefore \quad b = a \tan \text{ABC}$$

and

$$a = \frac{b}{\tan \text{ABC}}$$

From these results any one of the quantities, a, b, $\angle$ABC can be determined if the other two are known, since if we know the tangent it is easy to find the angle.

Similar relations can be determined for $\angle$BAC,

since $\quad \tan \text{BAC} = \frac{a}{b}$

4. Tangents of Angles less than 90°

Let a straight line OA, of **unit** length, rotate from a fixed position on OX so as to mark out 90° as shown in Fig. 66. Radiating lines are shown at 10°, 20°, 30°, etc.

If from any point B a perpendicular BN be drawn to OX, then the ratio $\frac{BN}{ON}$ is the tangent of the corresponding angle.

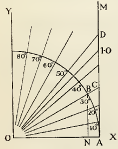


Fig. 66.

Let a perpendicular AM be drawn from A and the radial lines OB, etc., produced to meet it.

Then, considering one of the angles, BON,

$$\tan BON = \frac{CA}{OA}$$

Now, as OA is of **unit** length, the length of CA, on the scale selected, will give the actual value of the tangent of the corresponding angle COA. Similarly the tangents of other angles 10°, 20°, etc., can be read off by measuring the corresponding intercept on AM: *e.g.* the tangent of 50° is given by the length of AD.

From examination of these and similar results we may conclude:

(1) When the angle is 0°, tan 0° is 0.

(2) As the angle increases, tan θ increases.

(3) tan 45° = 1.

(4) For angles greater than 45° the tangent is greater than 1.

(5) As the angle approaches 90°, the tangent rapidly increases. When it is nearly 90°, the tangent is very great. This we usually express by saying that

As θ approaches 90°, tan θ approaches infinity.

5. The Tangent Graph

Using the values of the tangents of various angles by means of Fig. 61 or from tables as will be shown later, a graph may be drawn of the tangents of angles in the first quadrant. This is shown in Fig. 67.

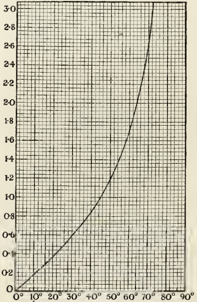


Fig. 67.

It will be seen that as the angle approaches 90°, the curve rises rapidly. We may say that it will meet the perpendicular from OX at 90° at an infinite distance.

6. A Table of Tangents

The tangents of angles can be obtained by the graphic methods used above. In practice this would be cumbersome and not very accurate. However, by using methods which will be studied in more advanced mathematics, the values of these tangents can be calculated to any required degree of accuracy. Tables thus obtained, correct to four places of decimals, are given at the end of this book, and these can be used when required in problems.

NATURAL TANGENTS

Degrees.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
											1	2	3	4	5
25	0·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19

A portion of the tables is shown above, giving tbe tangents of angles between 25° and 29°. The tangent of an angle with an exact number of degrees is shown in the first column; thus $\tan 27° = 0{\cdot}5095$. If the angle involves minutes as well as degrees, we use the other columns.

Thus tan 25° 24′ will be found under the column headed 24′. Thus $\tan 25° 24' = 0{\cdot}4748$.

If the number of minutes is not an exact multiple of 6, we use the column of mean differences for any number over a multiple of 6.

Thus to find tan 26° 38′.

tan 26° 36′ is 0·5008.

For 38′, *i.e.* 2 minutes over 36′, we turn to the mean difference column for 2 and we find 7. This is added on to tan 26° 36′ and so we get

$$\tan 26° 38' = 0{\cdot}5008 + 7$$
$$= 0{\cdot}5015$$

NOTE.—The mean differences are not sufficiently accurate to be useful when the angle is large, *e.g.* beyond 74°. In such cases a larger volume of tables must be consulted.

7. Slope and Gradient of a Path

On p. 199 we explained what is meant by the angle of slope or briefly the slope of a path. We can now deal with this more fully.

Fig. 68 represents the side view of a path AC, AB being

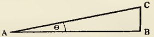


FIG. 68.

the horizontal. BC is perpendicular to AB. Then $\angle CAB$ is the angle of slope of the path.

$$\therefore \quad \tan CAB = \frac{CB}{AB}$$

This tangent is called the gradient of the path.

Generally—*If* θ *be the slope of the path, tan* θ *is the gradient.*

NOTE.—If the angle of slope be very small, AC is very little different in length from AB.

Consequently the ratio $\frac{CB}{AC}$ does not differ appreciably from $\frac{CB}{AB}$.

For further consideration of this, see p. 199.

8. The Equation of a Straight Line

In Chapter 7, p. 145, it was shown that the equation of a straight line is given in general terms by

$$y = mx + b$$

It is now possible to give a meaning to the constant *m*.

Consider the case of a straight line represented by the equation

$$y = 2x \quad \text{(Fig. 69)}$$

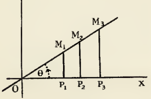


FIG. 69.

Let any points M_1, M_2, M_3 . . . be taken on the line and perpendiculars M_1P_1, M_2P_2, M_3P_3 . . . be drawn to the x axis.

Then, as we have seen (p. 222), the ratios

$$\frac{M_1P_1}{OP_1}, \quad \frac{M_2P_2}{OP_2}, \quad \frac{M_3P_3}{OP_3} \ . \ . \ .$$

all are equal and their common value is 2.

Also each ratio represents tan θ, where θ is the angle made by the straight line with the OX.

Generally if the co-ordinates of any point on the line are represented by (x, y), then

$$\frac{y}{x} = \tan \theta$$

or $$y = \tan \theta \times x$$

Comparing this with $y = mx$, it is clear that *m represents tan θ*, i.e. *the tangent of the angle which the straight line makes with the axis of x.*

m or tan θ is called the gradient of the line.

The gradient of the line is the tangent of the angle of slope.

If we consider a straight line parallel to $y = 2x$ (Fig. 70), and passing through the point $+ 3$ on the y axis, its equation, as shown on p. 000, is $y = 2x + 3$.

It obviously makes the same angle θ with the x-axis as $y = 2x$ and has the same gradient—*i.e.* tan θ = 2.

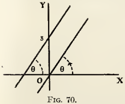


FIG. 70.

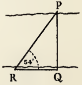


FIG. 71.

9. Examples of the Uses of Tangents

1. P and Q (Fig. 71) are two points on opposite sides of a river. A point R is 180 yd along the bank from Q. The angle PQR is a right angle and the angle PRQ is found to be 54°. Find the distance PQ. The surveying instruments which determine the angle 54° can be used to verify that PQR is a right angle.

From the diagram $$\frac{PQ}{RQ} = \tan 54°$$

$$\therefore \quad PQ = RQ \tan 54°$$
$$= 180 \times 1{\cdot}3764$$
$$= 248 \text{ yd approximately.}$$

2. From the top of a cliff, 250 ft high, the angle of depression of a boat on the sea was found to be 10° 30′. How far was the boat from the foot of the cliff?

If in Fig. 72 PQ represents the cliff and S the position of the boat, then ∠SPA made by PS with the horizontal is the angle of depression of the boat.

Then $\widehat{QPS} = 90° - \widehat{APS}$
$= 90° - 10° 30'$
$= 79° 30'$

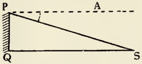


FIG. 72.

Also $\frac{SQ}{QP} = \tan QPS$

$\therefore \frac{SQ}{250} = \tan 79° 30'$

$\therefore$ $SQ = 250 \tan 79° 30'$
$= 250 \times 5{\cdot}3955$
$= 1350$ ft approximately.

10. The Sine and Cosine Ratios

Fig 73. represents an angle ABC (θ) and AC is perpendicular to BC. Then $\tan \theta = \frac{AC}{BC}$.

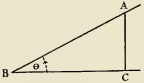


FIG. 73.

The ratios of AC and BC to the hypotenuse AB give us two other *constant ratios* for an angle, by each of which the angle may be identified.

The ratio $\frac{AC}{AB}$ is called the *sine* of the angle

" " $\frac{BC}{AB}$ " " *cosine* " "

They are abbreviated as follows:

$$\sin \theta = \frac{AC}{AB}$$

$$\cos \theta = \frac{BC}{AB}$$

It should be noted that since AB is the greatest side of the triangle ABC, the *sine and cosine can never be greater than unity.*

Ratios of complementary angles

From Fig. 68 $\sin BAC = \frac{BC}{AB} = \cos ABC$

$\cos BAC = \frac{AC}{AB} = \sin ABC$

Now, ABC and BAC are complementary angles (see Chapter 9, p. 198).

Hence *the sine of an angle is equal to the cosine of its complement, and the cosine of an angle is equal to the sine of its complement.*

This may be expressed $\sin \theta = \cos (90° - \theta)$
$\cos \theta = \sin (90° - \theta)$

11. The Sine in Mechanical Engineering

In the examples on heights and distances in the solution of which the tangent tables were employed it was the base of the standard right-angled triangle which could easily be measured by laying a tape measure (or a surveyors' chain) along the surface of the ground. It was assumed that the

angles involved could be accurately measured, and this could in fact be accomplished by the use of the theodolite or other surveying instrument embodying a sighting telescope. Thus heights and distances inaccessible for direct measurement could be determined. The user was, however, dependent upon the maker of the instrument for the accuracy of measurement of the angles whose tangents could then be read from the tables.

In mechanical engineering, however, the *lengths* concerned can generally be precisely determined—by means of

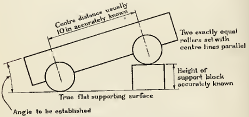


FIG. 74.—Principle of Engineers' Sine Bar.

micrometer screw gauges or vernier callipers (specimens of which can be found in any science laboratory). Angles can thus be measured with great accuracy, or an angle in the solid constructed to any desired size, by means of accurate measurements of length. Since all the sides of the standard right-angled triangle are accessible there is no need to concentrate on the base of this triangle or to use the tangent tables. In practice, the "opposite" side, and the "hypotenuse" or diagonal are measured and the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is calculated. Reference to the *sine* tables will then give the value of the angle between the base and the hypotenuse.

The diagram, Fig. 74, illustrates the principle of the "sine bar" used in engineering for constructing actual (in-the-solid) angles to correspond with the angles figured on drawings.

12. Sines of Angles in the First Quadrant

Fig. 75 represents a series of angles formed in the first quadrant by a line rotating from OA.

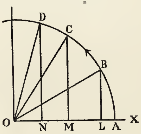


FIG. 75.

The *sines* of the angles are given by

$$\sin AOB = \frac{BL}{OB},$$

$$\sin COA = \frac{CM}{OC},$$

$$\sin DOA = \frac{DN}{OD}.$$

As the lengths of the denominators are equal and the numerators are increasing, it will be seen that:

(1) $\sin 0^\circ = 0$.
(2) *as* θ *increases* from 0° to 90° $\sin \theta$ *increases*.
(3) $\sin 90^\circ = 1$.

13. Cosines of Angles in the First Quadrant

Using Fig. 75, the cosines of the angles are given by

$$\cos AOB = \frac{OL}{OB}, \cos AOC = \frac{OM}{OC}, \cos AOD = \frac{ON}{OD}.$$

As before, the denominators of these are equal and the numerators this time are decreasing.

Hence we conclude:

(1) $\cos 0^\circ = 1$.
(2) *as* θ *increases* from 0° to 90° *cos* θ *decreases.*
(3) $\cos 90^\circ = 0$.

14. Tables

As with tangents, tables of the sines and cosines have been calculated and will be found at the end of this book. They are used in the same way as those of tangents, with the exception that since *cosines decrease as the angles increase*, as shown above, *mean differences must be subtracted instead of added.*

15. Graphs of Sin θ and Cos θ

By using tables or otherwise to obtain the sines and cosines of angles, the graphs of these may be drawn. They are as shown below, Figs. 76 and 77.

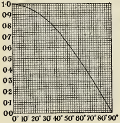


Fig. 76.

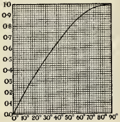


Fig. 77.

It will be seen that they are the same curves but differently placed. They are drawn for angles in the first quadrant only. The ratios of angles greater than these are dealt with in the second-year course. They present difficulties not found with angles in the first quadrant.

16. Cosecant, Secant and Cotangent

The reciprocals of the sine, cosine and tangent furnish us with three other ratios which are very useful in solving certain problems. They are named as follows:

$\frac{1}{\sin \theta}$ is called the cosecant (cosec θ).

$\frac{1}{\cos \theta}$ is called the secant (sec θ).

$\frac{1}{\tan \theta}$ is called the cotangent (cot θ).

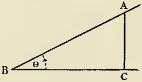


Fig. 78.

Thus in Fig. 78 with the usual construction

$$\frac{AC}{AB} = \sin \theta; \quad \frac{AB}{AC} = \text{cosec } \theta,$$

$$\frac{BC}{AB} = \cos \theta; \quad \frac{AB}{BC} = \sec \theta,$$

$$\frac{AC}{BC} = \tan \theta; \quad \frac{BC}{AC} = \cot \theta.$$

It should be noted that as the sine and cosine can never be greater than unity, the cosecant and secant can never be less than unity.

Tables of these ratios will be found in Books of Tables, and the method of using is similar to that of the other ratios.

17. Relations between the Trigonometrical Ratios

(1) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

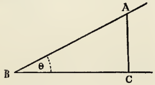


FIG. 79.

Let ABC (Fig. 79) be any angle (θ).
Let AC be perpendicular to BC.
As shown above

$$\sin\theta = \frac{AC}{AB}$$

$$\cos\theta = \frac{BC}{AB}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{AC}{AB} \div \frac{BC}{AB}$$

$$= \frac{AC}{AB} \times \frac{AB}{BC}$$

$$= \frac{AC}{BC}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \tan\theta \quad . \quad . \quad . \quad . \quad (1)$$

(2) $\sin^2\theta + \cos^2\theta = 1$

Using Fig. 79 and applying the Theorem of Pythagoras we have $\qquad AC^2 + CB^2 = AB^2$

Dividing by AB^2, $\dfrac{AC^2}{AB^2} + \dfrac{CB^2}{AB^2} = 1$

or $\qquad (\sin\theta)^2 + (\cos\theta)^2 = 1$

This is usually written

$$\sin^2\theta + \cos^2\theta = 1 \quad . \quad . \quad . \quad . \quad (2)$$

From this we can get

$$\sin\theta = \sqrt{1 - \cos^2\theta} \quad . \quad . \quad . \quad (3)$$

and

$$\cos\theta = \sqrt{1 - \sin^2\theta} \quad . \quad . \quad . \quad (4)$$

Thus we can find $\sin\theta$ when $\cos\theta$ is known and vice versa.

(3) $\qquad 1 + \tan^2\theta = \sec^2\theta$
$\qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Using the formula proved in (2), viz.

$$\sin^2\theta + \cos^2\theta = 1 \quad . \quad . \quad . \quad (2)$$

Divide by $\cos^2\theta$

then $$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

or $$\tan^2\theta + 1 = \sec^2\theta \quad . \quad . \quad . \quad . \quad (5)$$

Dividing (2) by $\sin^2\theta$

then $$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad . \quad . \quad . \quad (6)$$

Example. *The sine of an angle is* 0·8. *Find the cosine and tangent.*

By formula (3) above
$$\cos\theta = \sqrt{1-\sin^2\theta}$$
$$= \sqrt{1-(0{\cdot}8)^2}$$
$$= \sqrt{0{\cdot}36}$$
$$= 0{\cdot}6$$

Also
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad . \quad . \quad . \quad . \quad (1)$$
$$= \frac{0{\cdot}8}{0{\cdot}6}$$
$$= \frac{4}{3}$$

These simple ratios arise from the fact that sides 3, 4, and 5, build up a right-angled triangle.

18. Solution of Right-angled Triangles

If certain sides and angles of a triangle are given, it is possible to find the remaining sides and angles. This is called *solving the triangle.*

To solve a right-angled triangle we use the trigonometrical ratios and the theorem of Pythagoras.

The methods are indicated in the following examples.

Example 1. *Solve the right-angled triangle in which the sides which contain the right angle are* 15·8 *in. and* 8·9 *in.*

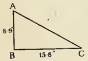


FIG. 80.

Using Fig. 80, we see that we require to find the angles at A and C and the hypotenuse.

(1) Using the theorem of Pythagoras, we find the hypotenuse. Thus

$$AC = \sqrt{15{\cdot}8^2 + 8{\cdot}9^2}$$

whence
$$AC = 18{\cdot}1 \text{ approx.}$$

(2) To find A and C we use the tangent ratio

$$\tan ACB = \frac{8{\cdot}9}{15{\cdot}8} \qquad \tan BAC = \frac{15{\cdot}8}{8{\cdot}9}$$
$$= 0{\cdot}5633 \qquad = 1{\cdot}7747$$
$$= \tan 29^\circ\, 24' \qquad = \tan 60^\circ\, 36'$$

Check —— $29^\circ\, 24' + 60^\circ\, 36' = 90^\circ$

Example 2. *Given the hypotenuse and one angle, solve the right-angled triangle in which the hypotenuse is* 6·85 *in. and one angle is* 27° 43′.

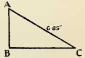


FIG. 81.

In Fig. 81 $\angle ACB = 27^\circ\, 43'$

Then $\angle CAB = 90^\circ - 27^\circ\, 43'$
$= 62^\circ\, 17'$

To find CB and AC use the cosine and sine ratios

$$CB = AC \cos ACB \qquad AB = AC \sin ACB$$
$$= 6{\cdot}85 \times \cos 27^\circ\, 43' \qquad = 6{\cdot}85 \sin 27^\circ\, 43'$$
$$= 6{\cdot}06 \text{ in.} \qquad = 3{\cdot}19 \text{ in.}$$

Example 3. *To solve the triangle given one angle and the sides containing the right angle.*

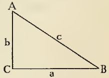


FIG. 82.

Using Fig. 82, in which the sides opposite the angles A, B, C are denoted by a, b, c respectively.

Suppose we know B *and* a

then
$$\frac{b}{a} = \tan B$$
$$\therefore \quad b = a \tan B$$

Also
$$\frac{a}{c} = \cos B \text{ or } \frac{c}{a} = \sec B$$
$$\therefore \quad c = a \sec B$$

Finally
$$A = 90^\circ - B$$

Example 4. *Equilateral Triangle.*

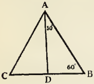


FIG. 83.

In the equilateral triangle, Fig. 83, draw AD perpendicular to the base.

It bisects the base and the angle at A.

Let each side of the triangle be a. Each of the angles is 60°.

In $\triangle ABD$, $AD^2 = AB^2 - BD^2$ (Theorem of Pythagoras)

$$= a^2 - \frac{a^2}{4}$$

$$= \frac{3a^2}{4}$$

$$\therefore \quad AD = a \times \frac{\sqrt{3}}{2}$$

Hence $\quad \sin 60^\circ = \dfrac{AD}{AB} = \dfrac{a \cdot \frac{\sqrt{3}}{2}}{a} = \dfrac{\sqrt{3}}{2}$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2} \div a = \tfrac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{a\sqrt{3}}{2} \div \frac{a}{2} = \sqrt{3}$$

Similarly

$$\sin 30^\circ = \tfrac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

Example 5. *Isosceles right-angled triangle.*

Fig. 84 represents a right-angled triangle in which AC = CB.

Then the angles at A and B are each 45°.

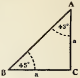


FIG. 84.

Let each of the equal sides be a

then $\quad AB^2 = AC^2 + CB^2$ (Theorem of Pythagoras)

$\therefore \quad AB^2 = 2a^2$

and $\quad AB = a\sqrt{2}$

Hence $\quad \sin 45^\circ = \sin ABC = a \div a\sqrt{2} = \dfrac{1}{\sqrt{2}}$

$$\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = 1$$

Example 6. *A 10-in. sine bar has its left-hand roller resting upon a block 0·2 in. thick. The right-hand roller rests upon a block 0·325 in. thick. Both blocks rest upon a plane horizontal surface. What angle does the upper surface of the bar make with the plane surface?* See Fig. 74.

In the standard right-angled triangle the side *opposite* the acute angle under consideration is (0·325–0·2) in. The hypotenuse is 10 in. So if the acute angle in question is θ,

$$\sin \theta = \frac{0{\cdot}125 \text{ in.}}{10 \text{ in.}} = 0{\cdot}0125$$

From the table of natural sines,

$$\sin 0^\circ\, 42' = 0{\cdot}0122$$

and 0·0003 is the difference for **1′**.

Therefore $\quad \theta = 0^\circ\, 43'$

19. Solution of triangles not right-angled. The Sine Rule

Let ABC, Fig. 85, be any acute-angled triangle. Denote sides by a, b, c, as previously indicated.

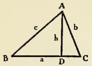


FIG. 85.

Draw AD perpendicular to BC.

In $\triangle$ACD, AD $= b \sin$ C.

In $\triangle$ABD, AD $= c \sin$ B.

$\therefore \quad b \sin \mathrm{C} = c \sin \mathrm{B}$

or $\quad \dfrac{b}{c} = \dfrac{\sin \mathrm{B}}{\sin \mathrm{C}}$

In a similar way we may show that $\dfrac{a}{b} = \dfrac{\sin \mathrm{A}}{\sin \mathrm{B}}$ or we may write the results in the form

$$\frac{a}{\sin \mathrm{A}} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}}$$

This is called the *Sine Rule* and it may be expressed thus:

The sides of a triangle are proportional to the sines of the opposite angles.

Example. *Solve the triangle in which* A $= 52° \ 15'$, B $= 70° \ 26'$, $a = 9{\cdot}8$ in.

NOTE.—In this and following examples the notation previously explained will be employed.

(1) *To find* C

$$\begin{aligned} \mathrm{C} &= 180° - (\mathrm{A} + \mathrm{B}) \\ &= 180° - (52° \ 15' + 70° \ 26') \\ &= 57° \ 19' \end{aligned}$$

(2) *To find b and c*

Using the sine rule,

$$\therefore \quad b = \frac{a \sin \mathrm{B}}{\sin \mathrm{A}}$$

$$= \frac{9{\cdot}8 \times \sin 70° \ 26'}{\sin 52° \ 15'}$$

$$\begin{aligned} \therefore \quad \log b &= \log 9{\cdot}8 + \log \sin 70° \ 26' - \log \sin 52° \ 15' \\ &= 0{\cdot}9912 + \bar{1}{\cdot}9742 - \bar{1}{\cdot}8980 \\ &= 0{\cdot}9654 - \bar{1}{\cdot}8980 \\ &= 1{\cdot}0674 \\ &= \log 11{\cdot}68 \end{aligned}$$

$\therefore \quad b = 11{\cdot}7$ to three significant figures.

Similarly by using $\quad \dfrac{c}{a} = \dfrac{\sin \mathrm{C}}{\sin \mathrm{A}}$

we find $\quad c = 10{\cdot}4$

$\therefore$ *the solution is* C $= 57° \ 19'$, $b = 11{\cdot}7$, $c = 10{\cdot}4$.

20. Area of a Triangle

One method of finding the area of a triangle has been shown on p. 30. There it was shown that if we have a triangle such as ABC (Fig. 86) and AD be drawn perpendicular to BC, then the area is $\frac{1}{2}$CB . AD.

Or with the usual notation, if h be the length of AD then area $= \frac{1}{2}ah$.

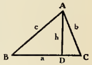


FIG. 86.

Sine formula

In the triangle ABD

$$\sin \mathrm{B} = \frac{h}{c}$$

$$\therefore \quad h = c \sin \mathrm{B}$$

Substituting this for h in the above formula

$$area = \tfrac{1}{2}ac \sin \mathrm{B}$$

Similarly

$$\begin{aligned} area &= \tfrac{1}{2}bc \sin \mathrm{A} \\ &= \tfrac{1}{2}ab \sin \mathrm{C} \end{aligned}$$

This rule may be expressed thus:

The area of a triangle is equal to one-half of the product of any two sides and the sine of the angle between them.

Worked Examples

Example 1. *A triangle* ABC *has its sides* $a = 10$, $b = 9$, $c = 8$. *A perpendicular* AD *is drawn from* A *to* BC. *Find the lengths of* BD *and* CD.

Let $CD = x$. Then $BD = 10 - x$ (Fig. 87).

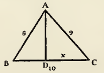


FIG. 87.

now $AD^2 = 8^2 - (10 - x)^2$ (Theorem of Pythagoras)

also

$$AD^2 = 9^2 - x^2$$

$$\therefore \quad 8^2 - (10 - x)^2 = 9^2 - x^2$$

$$64 - 100 + 20x - x^2 = 81 - x^2$$

$$\therefore \quad 20x = 117$$

and

$$x = \tfrac{117}{20}$$

$$\therefore \quad BD = 10 - x = 10 - \tfrac{117}{20}$$

$$= \tfrac{83}{20}$$

$\therefore$ the two parts are $5\frac{17}{20}$ and $4\frac{3}{20}$.

NOTE.—It should be observed that with these results we could find cos C and cos B and so solve the triangle.

Example 2. A *and* B *are two points on the same horizontal level and* 1200 *yd apart. An aeroplane is due east of them, and the angles of elevation from* A *and* B *are* 52° *and* 70°. *Find the distances of the aeroplane from* A *and* B.

Let P represent the position of the aeroplane (Fig. 88). Draw PQ perpendicular to AB produced.

We require to find AP and BP.

This can be done if we know BQ.

Let $BQ = x$. Then $AQ = 1200 + x$

In $\triangle PBQ$, $\dfrac{PQ}{BQ} = \tan 70°$ or $PQ = x \tan 70°$.

In $\triangle PAQ$, $\dfrac{PQ}{AQ} = \tan 52°$

$$\therefore \quad PQ = (1200 + x) \tan 52°$$

$$\therefore \quad x \tan 70° = (1200 + x) \tan 52°$$

$$\therefore \quad x(\tan 70° - \tan 52°) = 1200 \tan 52°$$

$$\therefore \quad x = \frac{1200 \tan 52°}{\tan 70° - \tan 52°}$$

$$= \frac{1200 \times 1{\cdot}2799}{2{\cdot}7475 - 1{\cdot}2799}$$

$$= \frac{1536}{1{\cdot}468} = 1046 \text{ yd. approx.}$$

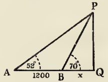


FIG. 88.

Now $\dfrac{BP}{x} = \sec 70°$

$$\therefore \quad BP = x \sec 70° = 1046 \times 2{\cdot}9238$$

$$= 3058 \text{ yd. approx.}$$

Similarly $AP = AQ \sec 52°$

$$= (1200 + 1046) \sec 52°$$

$$= 2246 \times 1{\cdot}6243$$

$$= 3648 \text{ yd. approx.}$$

EXERCISE X

Section A. The Tangent

1. In Fig. 89 ABC is a right-angled triangle with C the right angle. Draw CD perpendicular to AB and DQ perpendicular to CB.

Write down the tangents of ABC and CAB in as many ways as possible using lines of the figure.

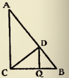


Fig. 89.

2. Construct an angle whose tangent is 0·6 and measure the angle.

3. In Fig. 89 if AB is 15 cm and AC 12 cm in length, find the values of tan ABC and tan CAB.

4. From the tables write down the tangents of the following angles:

(1) 18°	(4) 73°
(2) 43°	(5) 14° 18′
(3) 56°	(6) 34° 48′

5. Write down the tangents of:

(1) 9° 17′	(4) 52° 27′
(2) 31° 45′	(5) 64° 40′
(3) 39° 5′	

6. From the tables find the angles whose tangents are:

(1) 0·5452	(4) 1·3001
(2) 1·8265	(5) 0·6707
(3) 2·8239	(6) 0·2542

7. When the altitude of the sun is 48° 24′, find the height of a flagstaff whose shadow is 26 ft 6 in. long.

8. Find the vertical angle of a cone in which the diameter of the base is 10·6 in. and the height is 12·4 in.

9. The base of an isosceles triangle is 10 in. and each of the equal sides is 13 in. Find the angles of the triangle.

10. A ladder rests against the top of the wall of a house and makes an angle of 69° with the ground. If the foot is 20 ft from the wall, what is the height of the house?

11. From the top window of a house which is 75 yd away from a tower it is observed that the angle of elevation of the top of the tower is 36° and the angle of depression of the bottom is 12°. What is the height of the tower?

12. From the top of a cliff 320 ft high it is noted that the angles of depression of two boats lying in a line due east of the cliff are 21° and 17°. How far are the boats apart?

13. A regular hexagon circumscribes a circle of 10 in. radius. Find the perimeter of the hexagon.

14. The angle of elevation of the top of a vertical tower at a horizontal distance of 100 ft from the foot of the tower is 56°. Calculate (i) the height of the tower, (ii) the angle of elevation of the top of the tower from a point whose horizontal distance from the tower is 200 ft and whose height above the horizontal plane through the foot of the tower is 55 ft. (N.C.T.E.C.)

15. Two adjacent sides of a rectangle are 15·8 cm and 11·9 cm. Find the angles which a diagonal of the rectangle makes with the sides.

Section B. The Sine and Cosine

1. From Fig. 89 write down in as many ways as possible the sine and cosine of ABC and CAB, using the lines of the figure.

2. Draw a circle with radius 1·5 in. Draw a chord of length 2 in. Find the sine and cosine of the angle subtended by this chord.

3. In a circle of 4 in. radius a chord is drawn subtending an angle of 80° at the centre. Find the length of the chord and its distance from the centre.

4. The sides of a triangle are 4·5 in., 6 in. and 7·5 in. Draw the triangle and find the sines and cosines of the angles.

5. From the tables write down the sines of the following angles:

(1) 14° 36′ (2) 47° 44′ (3) 69° 17′

6. From the tables write down the angles whose sines are:

(1) 0·4970 (2) 0·5115 (3) 0·7906

7. From the tables write down the cosines of the following angles:

(1) 20° 46′	(4) 38° 50′
(2) 44° 22′	(5) 79° 16′
(3) 62° 39′	(6) 57° 23′

8. From the tables write down the angles whose cosines are:

(1) 0·5332	(4) 0·2172
(2) 0·9358	(5) 0·7910
(3) 0·3546	(6) 0·5140

9. A certain uniform incline rises 10 ft 6 in. in a length of 60 ft along the incline. Find the angle between the incline and the horizontal.

Find also the rise of an incline of 100 ft long which makes an angle of 20° with the horizontal. (U.L.C.I.)

10. Construct an angle whose cosine is three times its sine. Measure it and check your result from the tables.

11. In a right-angled triangle the sides containing the right angle are 4·5 in. and 5·8 in. Find the angles and length of the hypotenuse.

12. Draw an angle whose tangent is 0·75 and find its sine and cosine. (U.E.I.)

Section C. Secant, Cosecant and Cotangent

1. From the tables find the following:

(1) cosec 35° 24′	(4) sec 53° 5′
(2) cosec 59° 45′	(5) cot 39° 42′
(3) sec 42° 37′	(6) cot 70° 34′

2. From the tables find the angles:

(1) whose cosecant is 1·1476.
(2) whose secant is 2·3443.
(3) whose cotangent is 0·3779.

3. The height of an isosceles triangle is 3·8 in. and each of the equal angles is 52°. Find the lengths of the equal sides.

4. Construct a triangle with sides 5 cm, 12 cm and 13 cm in length. Find the cosecant, secant and tangent of each of the acute angles. Hence find the angles from the tables.

5. A chord of a circle is 3 in. long and it subtends an angle of 63° at the centre. Find the radius of the circle.

Section D. Right-angled Triangles

1. In a right-angled triangle the two sides containing the right angle are 23·4 in. and 16·4 in. Find the angles and the hypotenuse.

2. ABC is a triangle, C being a right angle.

If A = 51° 7′, and $b = 36{\cdot}64$, find a and c.

3. ABC is a right-angled triangle, C being the right angle. If $a = 378$ ft and $c = 543$ ft, find A and b.

4. A ladder 20 ft long rests against a vertical wall. By means of trigonometrical tables find the inclination of the ladder to the horizontal when the foot of the ladder is:

(1) 7 ft from the wall.
(2) 10 ft from the wall.

Use these angles to calculate how far the top of the ladder descends when the ladder is moved from its first to its second position. (N.C.T.E.C.)

5. AB, AC are the two legs of a pair of "steps."

AB = 10 ft, AC = 12·5 ft.

The steps are set up on level ground with the leg AB inclined at 76° to the horizontal. Calculate (i) the height of A above the ground, (iii) the angle ACB.

(N.C.T.E.C.)

6. Calculate in square inches to three significant figures the area of the largest hexagon which can be cut out of a circular plate of diameter $9\frac{1}{2}$ in. (U.E.I.)

7. (*a*) A pendulum of length 20 cm swings through an angle of 15° on either side of the vertical. Through what height does the bob rise?

(*b*) If $\cos A = \frac{8{\cdot}72}{9{\cdot}83} \sin 23°$, calculate the angle A to the nearest degree. (U.E.I.)

8. P and Q are points on a straight coast-line, Q being 5·3 miles east of P. A ship starting from P steams 4 miles in a direction $65\frac{1}{2}°$ north of east. Calculate:

(i) the distance the ship now is from the coast-line.
(ii) the ship's bearing from Q. (N.C.T.E.C.)

Section E. Solution of a Triangle and Miscellaneous

1. The left-hand roller of a 5-in. sine bar rests directly upon a true surface plate. To what heights above the surface plate must the right-hand roller be packed up in order that the bar may make angles of 20°, 30°, 45° respectively above the surface plate? Note that packings whose dimensions are known accurately to 0·0001 in. are available in engineering workshops.

Why would it not be good practice to set the sine bar at say 75° with the surface plate?

2. For a ball $1\frac{1}{4}$ in. dia resting in a 65° V calculate the distance from the apex of the V to a point of contact of the ball.

(i) If $\sin A = \frac{21}{29}$, find, by sketching a right-angled triangle ABC, the values of cos A and tan A. If BD is the perpendicular from B on to AC, find cos ABD.
(ii) If $\cos \theta = 0{\cdot}8878$, find $\sin \theta$ and $\tan 2\theta$.
(iii) Find values of x, between 0 and 360°, which satisfy the equation $2{\cdot}5 \sin x = 1{\cdot}5 \cos x$.

(Sunderland.)

3. The angle of elevation of the top of a tower from a point A on the ground is 48°. The angle of depression from an observation balloon 1000 ft vertically above A is 40°. Find (i) the height of the tower, (ii) the elevation of the balloon from the base of the tower. (Sunderland.)

4. If $\cos A = \frac{8{\cdot}72}{9{\cdot}83} \sin 23°$ calculate the angle A to the nearest degree. (Sunderland.)

5. A steel roller 6 ft in diameter is pushed between two steel plates hinged together at one end. What will be the angle between the plates when the roller axle is 15 ft from the hinge? (Nuneaton.)

6. A 90° notch is machined in a metal block, its section having sides 0·3 in. and 0·4 in. long. How large a cylinder can lie in the notch with its upper surface level with the upper surface of the block? If this calculation should prove too difficult you may proceed by drawing.

7. Find the length of the open belt connecting wheels of radii 2 ft and 6 in., placed with their centres 6 ft apart, and in a common plane.

If the larger wheel is rotating at 200 r.p.m., find the speed of any point P on the rim of the smaller wheel in feet per second. (Take $\pi = 3{\cdot}142$.) (Sunderland.)

8. (*a*) Construct an acute angle whose tangent is $\frac{12}{5}$.

From your diagram or otherwise determine the sine and cosine of the angle.

(*b*) Prove that $\sin^2 x + \cos^2 x = 1$.

ABC in Fig. 90 represents a seam of coal. If A is at a depth of 1000 ft, determine to the nearest 10 ft the depth of the seam at C. (U.L.C.I.)

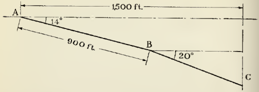


FIG. 90.

9. In a triangle ABC, AB = 3·5 in. and AC = 1·7 in. The length of the perpendicular from B on to AC produced is 1·4 in. Calculate the length of BC. (U.L.C.I.)

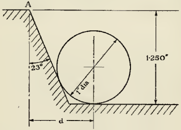


FIG. 91.

10. A plug of diameter 1 in. rests in an angle as shown in Fig. 91. The depth of the angle is 1·250 in. The sloping side of the angle makes 23° with the vertical and the other side is horizontal.

Calculate the horizontal distance d of the centre of the plug from the vertical through the point A. (Nuneaton.)

11. A circle is inscribed in an equilateral triangle of side 6 in. What is the radius of the circle? (W.R. Yorks.)

12. (a) The legs of a builder's trestle are 10 ft long. Calculate the angle between the legs when the feet are 5 ft apart. Also find the height of the trestle.

(b) Use logarithms to evaluate:

$$\text{(i)}\ \sqrt{101{\cdot}6^2 - 52{\cdot}3^2} \qquad \text{(ii)}\ \frac{1}{36{\cdot}8} + \frac{1}{0{\cdot}93}$$

(Worcester.)

13. Using your tables find (i) sin 120°; (ii) cos 237°; (iii) tan 307°. (W.R. Yorks.)

14. At 6 ft above the ground and at a horizontal distance of 45 ft from a pole the angle of elevation of the top of the pole is 36°. What is the height of the pole? (W.R. Yorks.)

15. A vertical cliff is 320 ft high. Calculate the angle of elevation of the top of the cliff from a boat which is $\frac{3}{8}$ mile out from the foot of the cliff. (Shrewsbury.)

16. (a) Given that $\sin\theta = \frac{4}{5}$, find the value of

$$\frac{\sin\theta - \cos\theta}{2\tan\theta}.$$

(b) Show that $(\sin A + \cos A)^2 = 2 \sin A \cdot \cos A + 1$.

(c) Find the value of $\sin(ft + c)$ when $f = 600$, $t = 0{\cdot}01$ and $c = -0{\cdot}2546$, the angle being expressed in radians.

(d) Solve the equation $10\sin^2\theta + 9\cos\theta = 12$ for values of θ between 0° and 360°. (Dudley.)

17. A swimming-bath is 30 ft wide and 66 ft long. The floor slopes uniformly at an angle of 15°. The depth at the shallow end is 3 ft 6 in. Calculate the volume of water in gallons required to fill it. (1 cu ft = 6·25 gal.) (Worcester.)

18. The sides of a triangle are 13 in., 8 in. and 10 in. long. A perpendicular is drawn to the largest side from the opposite angle. What angle does it make with the other sides?

19. In a survey a point C is observed from two other points A and B, 300 yd apart. The angles ABC and BAC are found to be 45° and 60° respectively. Calculate the length of AC and the shortest distance from C to AB. (U.L.C.I.)

20. In a reciprocating engine the lengths of the crank CA and the connecting-rod AB are 1 ft and 4·8 ft, respectively. Calculate the value of the angle ABC to the nearest degree, when the angle ACB is 86°. (U.L.C.I.)

21. Find the areas of triangles when:

(1) $b = 39{\cdot}6$ ft, $c = 50{\cdot}8$ ft, $A = 62° 37'$
(2) $a = 2{\cdot}9$ in., $b = 31{\cdot}5$ in., $C = 37° 28'$

22. The pitch diameter of a chain-wheel is given by

$$\frac{A}{\sin C}$$

where

$$\tan C = \frac{\sin\left(\frac{180°}{N}\right)}{\frac{B}{A} + \cos\left(\frac{180}{N}\right)^{\circ}}$$

Calculate the pitch diameter when $N = 15$, $B = 0{\cdot}564$ and $A = 0{\cdot}936$. (U.E.I.)

23. In connection with an engine governor, the following equation occurred:

$$\frac{\tan 30°}{\sin 30° + \frac{m}{b}} = \frac{\tan 45°}{\sin 45° + \frac{m}{b}}$$

Solve the equation for $\frac{m}{b}$. (U.E.I.)

CHAPTER 11

THE CIRCLE AND CIRCULAR MEASURE

1. Introduction to the Circle

If we examine a series of concentric circles it is evident that the length of the circumference must be definitely related to the length of the radius.

The calculation of this relation has been a problem in mathematics through the ages, but the exact method of obtaining it will be learnt by the student when he has made more progress in this work.

It can, however, be determined approximately by practical experiment.

For example, we know that the section of a cylinder at right angles to its axis is a circle.

Hence, by wrapping a piece of smooth paper tightly round a cylinder until it just overlaps, piercing the paper by a pin just beyond the overlapping and then opening out the paper, we can measure the distance between the two pinholes, and so get an approximate value for the circumference.

The **diameter** can be measured by callipers or other means.

By varying the method or by using other cylinders, it is found that the ratio of the circumference to the diameter gives a constant result, allowance being made for experimental error.

From experimental values it is found that this constant is approximately 3·14.

The ratio of the circumference to the diameter, how-

ever, is usually denoted by the Greek letter π, so that we have:

$$\frac{\mathbf{C}}{d} = \pi$$

or $$\mathbf{C} = \pi d$$

Taking r to represent the radius so that $d = 2r$, we have:

$$\mathbf{C} = 2\pi r$$

NOTE.—Though by experimental methods, the degree of accuracy obtainable for the value of π is limited, by higher mathematics we can calculate its value to any required degree of accuracy.

Thus to five decimal places $\pi = 3{\cdot}14159$.

This means $3{\cdot}1416$ to 5 significant figures or $3{\cdot}142$ to 4 figures.

For rough calculations it is sometimes taken as $\frac{22}{7}$ or $3\frac{1}{7}$.

2. Relation of Arc to Angle

Let the line OA rotate about O as centre, and trace out equal angles BOA and COB, Fig. 92.

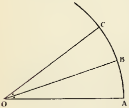


FIG. 92.

Clearly, then, the arc AB = the arc BC and arc AC = twice the arc AB or the arc BC.

But

$$\angle AOC = \text{twice } \angle AOB.$$

From this and similar examples it is evident that **the arc is proportional to the angle which it subtends at the centre.**

Now, the circumference subtends an angle of 360°.

If, then, θ degrees be the angle opposite any given arc of length a

$$\frac{\theta^\circ}{360^\circ} = \frac{a}{2\pi r}$$

From this $$a = \frac{2\pi r\theta^\circ}{360^\circ}$$

This enables us to find the length of an arc of a circle when we know the angle it subtends at the centre, and conversely.

Example 1. *How many revolutions are made by a 28-in. bicycle wheel in travelling $\frac{1}{2}$ mile?*

(Take $\pi = \frac{22}{7}$.)

Now $\quad C = \pi d$

that is $\quad C = \frac{22}{7} \times 28 = 88$ ins.

$$\therefore \text{ No. of revolutions} = \frac{880 \times 3 \times 12}{88}$$

$$= 360.$$

Example 2. *The circumference of the base of a cone of height 12 in. is 38 in. What is the circumference of a section parallel to the base and $3\frac{1}{2}$ in. from it?*

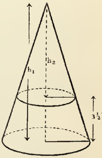


FIG. 93.

The section will be the base of a similar cone of height $8\frac{1}{2}$ in. (Fig. 93).

By similar figures the radii of the bases of the cones are proportional to the heights of the cones measured from the apex.

Let r_1 and r_2 be the radii of the bases, h_1 and h_2 the corresponding heights.

Then $$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$

and since the circumferences are proportional to the corresponding radii: If C_1 and C_2 represent the circumferences,

then $$\frac{h_1}{h_2} = \frac{C_1}{C_2}$$

$$\therefore \quad \frac{12}{8\frac{1}{2}} = \frac{38}{C_2}$$

that is, $$C_2 = \frac{38}{12} \times \frac{17}{2} = 26{\cdot}91 \text{ in.}$$

Example 3. *The end of a minute hand of a clock moves through 4·2 in. in 18 min. Find the length of the minute hand.*

The end of the hand traces out a circle, and in 18 min it rotates through an angle of $\frac{18}{60} \times 360 = 108°$.

$$\text{Since } \frac{\theta}{360} = \frac{a}{2\pi r}$$

$$\frac{108}{360} = \frac{4{\cdot}2}{2 \times 3{\cdot}142r}$$

$$\therefore \quad r = \frac{4{\cdot}2 \times 360}{108 \times 2 \times 3{\cdot}142}$$

$$= 2{\cdot}23 \text{ in. nearly.}$$

3. Area of a Circle

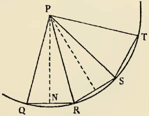


FIG. 94.

Area of a polygon.

Let QR be one side of a regular polygon inscribed in a circle, whose centre is P, and whose radius is r (Fig. 94). Draw PN perpendicular to QR. This will bisect QR, since PQR is an isosceles triangle. Now, the area of the triangle PQR $= \frac{1}{2}$PN $\times$ QR. Then the area of the whole polygon would be equal to the sum of the areas of the triangles such as PQR—that is, $\frac{1}{2}$PN(QR + RS + ST + . . .).

$\therefore$ Area of Polygon $= \frac{1}{2}$PN $\times$ sum of the sides.
$= \frac{1}{2}$PN $\times$ perimeter of polygon.

Area of a circle.

Now, the circle may be regarded as being made up of an infinitely large number of triangles, whose bases are infinitely small, so that if we increase the number of the sides of the polygon indefinitely, the perimeter of the polygon ultimately approaches closely to the circumference of the circle itself, and PN becomes a radius.

Hence area of circle $= \frac{1}{2}$ perimeter $\times$ radius.
$= \frac{1}{2} \times 2\pi r \times r.$
$= \pi r^2.$

Let $\quad$ A = the area.

Then $\quad \mathbf{A = \pi r^2}$

and $\quad \frac{A}{r^2} = \pi$ (a constant quantity).

Expressing this in words, we say that **the area of a circle is proportional to the square of its radius.** (See Chapter 9, pp. 211 and 212.)

4. Area of an Annulus or Circular Ring

An Annulus is a figure (Fig. 95) which is bounded by two concentric circles and its area is the difference between the areas of the two circles.

Let R and r be the radii of the outer and inner circles. Then area of Annulus

$= \pi R^2 - \pi r^2$
$= \pi(R^2 - r^2)$
$= \pi(R + r)(R - r)$
$= \pi$(**Sum of radii**) $\times$ (**Diff. of radii**).

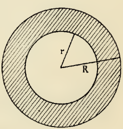


FIG. 95.

More frequently the diameter form is used, in which case the area is written in the form

$$\pi\left(\frac{D+d}{2}\right)\left(\frac{D-d}{2}\right)$$

$$= \frac{\pi}{4}(D + d)(D - d)$$

$$= 0{\cdot}7854\ (D + d)(D - d) \text{ approx.}$$

Written in this way the formula is suitable for the use of logarithms.

5. Area of a Sector of a Circle

Let PQR and PRS, Fig. 96, be sectors of a circle in which the chord QR = chord RS.

Since chord QR = chord RS

$\angle QPR = \angle RPS$

and $\angle QPS = 2\ \angle QPR$ or $2\ \angle RPS$.

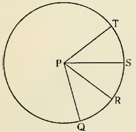


FIG. 96.

Clearly also the area of the sector PQR = area of sector RPS and ∴ area of sector QPR = twice area of sector QPR. Suppose an angle at the centre to be n times the $\angle QPR$. Then the area of the corresponding sector = n (area of sector QPR).

Hence the area of a sector of a circle is proportional to its angle at the centre.

It follows that $\dfrac{\text{Area of sector}}{\text{Area of circle}} = \dfrac{\text{Angle of sector}}{360°}$

Let $A \equiv$ the area of the sector

$\theta \equiv$ its angle at the centre.

Then $\dfrac{\mathbf{A}}{\pi r^2} = \dfrac{\theta}{360°}$

or $\mathbf{A} = \dfrac{\pi r^2 \cdot \theta}{360}$

Example 1. *The area of a circular pond is ¾ acre. Find its diameter to the nearest foot.*

$$A = \pi r^2$$

$$= \frac{\pi}{4}d^2 = 0{\cdot}7854 d^2$$

and ¾ acre = 32,670 sq ft.

$\therefore\ 0{\cdot}7854 d^2 = 32{,}670$

that is $d = \sqrt{\dfrac{32{,}670}{0{\cdot}7854}}$

Taking logs both sides

$$\log d = \tfrac{1}{2}\{\log 32{,}670 - \log 0{\cdot}7854\}$$

$$= \tfrac{1}{2}\{4{\cdot}5141 - \bar{1}{\cdot}8951\}$$

$$= 2{\cdot}3095$$

$$\therefore\ d = 203{\cdot}9$$

$$= 204 \text{ ft to the nearest foot.}$$

Example 2. *A circular path is 3 ft wide. If the inner boundary is a circle of 24 ft diameter, what would it cost to pave it at 7½d. per sq ft.*

The path is an annulus.

Let $A \equiv$ its area.

Then $A = \pi(R + r)(R - r)$

$= \frac{22}{7} \times (15 + 12)(3).$

$= \frac{22}{7} \times 27 \times 3$ sq ft.

$\therefore$ Cost $= \frac{22}{7} \times 27 \times 3 \times \frac{15}{2}$ pence

$= 1909$ pence to the nearest penny.

$= £7\ 19s.\ 1d.$

6. Determination of the Area of a Segment of a Circle

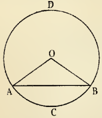


FIG. 97.

Let O be the centre of a circle and let the chord AB divide the circle into two segments ACB and ADB (Fig. 97). Join OA and OB.

Then

Area of segment ACB = Area of sector OAB — Area of △ OAB.

Example. *In a circle of radius 3·64 in. a chord is drawn which subtends an angle of 102° at the centre. What is the area of the minor segment?*

(1) Let A = the area of the sector OAB, Fig. 97.

$$\text{Now } A = \frac{\pi r^2 \times 102^\circ}{360^\circ}$$

$$= \frac{3{\cdot}142 \times 3{\cdot}64^2 \times 102}{360}$$

$$\begin{aligned}\text{Then } \log A &= \log 3{\cdot}142 + 2\log 3{\cdot}64 + \log 102 - \log 360\\ &= 0{\cdot}4972 + 1{\cdot}1222 + 2{\cdot}0086 - 2{\cdot}5563\\ &= 1{\cdot}0717\end{aligned}$$

$\therefore \quad A = 11{\cdot}80$ sq in.

(2) $$\begin{aligned}\text{Area of triangle AOB} &= \tfrac{1}{2} \times OA \times OB \sin AOB\\ &= 0{\cdot}5 \times 3{\cdot}64^2 \times \sin 102^\circ\\ &= 6{\cdot}480 \text{ sq in.}\end{aligned}$$

$\therefore$ **Area of segment** $\text{AOB} = 11{\cdot}80 - 6{\cdot}48 = 5{\cdot}32$ sq in.

7. Area of a Regular Figure Inscribed in a Circle

Suppose a figure of n equal sides to be inscribed in a circle of radius r.

Then the angle at the centre subtended by one of these sides is $\left(\frac{360}{n}\right)^\circ$.

If the angular points of the figure be joined to the centre, there will be n triangles.

The accompanying figure (Fig. 98) shows two of these triangles having $\angle AOB = \angle BOC = \left(\frac{360}{n}\right)^\circ$ and $OA = OB = OC = r$.

Now, the area of a triangle with two sides and the included angle given can be obtained from the formula $A = \frac{1}{2} ab \sin C$. (See p. 243.)

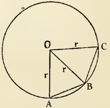


FIG. 98.

Hence area of △ AOB or △ BOC

$$= \tfrac{1}{2} r \times r \times \sin\left(\frac{360}{n}\right)$$

$$= \tfrac{1}{2} r^2 \sin\left(\frac{360}{n}\right)$$

$\therefore$ **Area of whole figure**

$$= \tfrac{1}{2} n r^2 \sin\left(\frac{360}{n}\right)^\circ.$$

Example. *Find the area of a regular pentagon inscribed in a circle of radius 2·64 in.*

Angle at centre subtended by one side $= \frac{360}{5} = 72^\circ$.

Then, using the formula above:

Area of one of the five triangles $= \frac{1}{2} \times 2{\cdot}64^2 \times \sin 72^\circ$

$\therefore$ Area of pentagon, $A = \frac{1}{2} \times 2{\cdot}64^2 \times \sin 72^\circ \times 5$

that is $A = 0{\cdot}5 \times 2{\cdot}64^2 \times 0{\cdot}9511 \times 5$

whence $\log A = 1{\cdot}2194$

$\therefore \quad A = 16{\cdot}58$ sq in.

Circular Measure

8. Hitherto in this volume the magnitude of an angle has been expressed in degrees or grades, which are obtained by the division of a right angle into an arbitrary number of parts.

This is the method in common use, and it originated in very early times in the history of the world.

There is another method, however, which is of great practical importance and in which the unit employed is an absolute one.

It can be explained as follows:

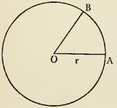


FIG. 99.

Suppose the line OA (Fig. 99) to rotate about the point O to the position OB, so that the length of the **arc AB** is equal to the radius of the circle. Then the angle AOB subtended by the arc AB is called a **radian. This angle is the unit of measurement in circular measure.** It is of constant size whatever the length of the radius.

Hence a **Radian** may be defined as **the angle subtended at the centre of a circle by an arc equal in length to the radius.**

The magnitude of an angle, expressed in Radians, is called the **Circular Measure** of that angle.

Length of an arc when the angle is given in radians.

Let θ radians be the angle subtended by an arc, and let r be the radius of the circle of which the arc forms a part.

Then, length of arc for 1 radian $= r$

$\therefore$ Length of arc for θ radians $= r\theta$

$$\therefore \quad \textbf{arc} = \mathbf{r\theta}.$$

9. Relation between Radians and Degrees

Since an arc r units in length subtends an angle of 1 radian, the number of radians subtended by the circumference of a circle is given by the number of times the *radius* is contained in the circumference.

Now, circumference $= 2\pi r$

Hence the number of radians for one revolution

$$= \frac{2\pi r}{r}$$

$$= 2\pi \text{ radians}$$

$$\therefore \quad 2\pi \text{ radians} = 360°$$

or π radians $= 180°$

that is $1 \text{ radian} = \dfrac{180}{\pi}$

$$= 57{\cdot}3° \text{ correct to 1 decimal place.}$$

Example. *Express an angle of* 113° 30′ *in radians.*

$$180° = \pi \text{ radians}$$

$$\therefore \quad 113{\cdot}5° = \frac{\pi \times 113{\cdot}5}{180}$$

$$= \frac{3{\cdot}142 \times 113{\cdot}5}{180}$$

$$= 1{\cdot}981 \text{ radians}$$

The following equivalents are worth remembering:

$$\theta° = \left(\frac{\pi}{180} \times \theta°\right) \text{ radians}$$

$$90° = \frac{\pi}{2} \text{ radians} \qquad 30° = \frac{\pi}{6} \text{ radians}$$

$$60° = \frac{\pi}{3} \text{ radians} \qquad 45° = \frac{\pi}{4} \text{ radians}$$

10. Relation between Angular and Linear Velocity

Let QMN (Fig. 100) represent a flywheel which has an angular velocity of ω (omega) radians per sec.

This means that any radius OQ rotates through an angle of ω radians in 1 sec.

Any point P on OQ will also have the same angular velocity.

Since arc $= r\theta$, the arc traced out by Q in 1 sec $= \omega$. OQ, and the arc traced out by P in 1 sec $= \omega$. OP.

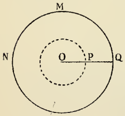


FIG. 100.

In general, if the point is at a distance r from the centre of rotation, the linear velocity of that point will be ωr.

Let $v \equiv$ the linear velocity of a point.

Then $v = \omega r$.

It should be carefully noted that though all points on the flywheel have the same angular velocity, the linear velocity of any point will depend on its distance from the centre of rotation.

Example. *A flywheel of radius 2 ft $4\frac{1}{2}$ in. is revolving at 80 revolutions per minute. Find the velocity in space of a point on its rim.*

Since 1 complete rotation is equivalent to 2π radians, the angular velocity of any point on its rim is $(80 \times 2\pi)$ radians per minute—that is $\dfrac{80 \times 2\pi}{60}$ radians per sec.

We have seen that $v = \omega r$, where ω is the angular velocity and r is the distance of any point from the centre of rotation. Hence the linear velocity of a point on the rim

$$\begin{aligned} &= \tfrac{80}{60} \times 2\pi \times 2{\cdot}375 \text{ ft per sec} \\ &= \tfrac{8}{6} \times 2\pi \times 2{\cdot}375 \text{ ft per sec} \\ &= 19{\cdot}9 \text{ ft per sec.} \end{aligned}$$

EXERCISE XI

At the discretion of the teacher, students may proceed directly to the miscellaneous exercises commencing on p. 270.

SECTION A

1. Find the circumferences of the circles whose radii are: (*a*) 5·6 in.; (*b*) 17·4 ft; (*c*) 2·9 cm.

2. Find the diameters of the circles whose circumferences are: (*a*) 370·4 ft; (*b*) 28·6 in.; (*c*) 15·2 cm.

3. A drain-pipe has a diameter of 3 ft 2 in. What is its circumference?

4. Taking the diameter of the earth as 7920 miles, what is its circumference?

5. If $r \equiv$ the radius and $\theta \equiv$ the angle subtended by an arc, find the length of the arc when (1) $r = 2{\cdot}5$ in., $\theta = 70°$; (2) $r = 11{\cdot}4$ cm, $\theta = 42\frac{1}{2}°$.

6. The path of a pen in a mechanism is an arc of a circle of 25 in. radius subtending an angle of 70° at the centre of the circle. Calculate:

   (1) the length of the path traversed by the pen;
   (2) the shortest distance between the two extreme positions of the pen. (U.E.I.)

7. A chord 1·8 in. long is drawn in a circle of radius 1·2 in. What are the lengths of the arcs into which the circumference is divided?

8. A thin steel band $\frac{3}{4}$ in. wide is fastened round a cylindrical tube of diameter $1\frac{3}{4}$ ft. What is the area of sheet-metal required?

9. How many revolutions will a wheel make in travelling $\frac{1}{2}$ mile if its diameter is $2\frac{1}{2}$ ft?

10. The length of a pendulum measured from its point of suspension to the lowest point is $2\frac{1}{2}$ ft. If in a swing from left to right it traces out an angle of 25°, over what distance does the lowest point travel?

11. Express algebraically in each case the difference between the circumferences of two circles when

(1) The radii are R in. and (R − 3) in.
(2) The diameters are D ft and d in. respectively.

SECTION B

1. Find the areas of the circles whose radii are: (*a*) 1·4 cm; (*b*) 3·8 in.; (*c*) 12·5 ft.

2. Find the radii of the circles whose areas are: (*a*) 12·6 sq in.; (*b*) 1250 sq ft; (*c*) 40 sq cm; (*d*) 32·79 sq in.

3. Find the circumference of a circle whose area is 64·75 sq in.

4. Find the circumference of a circle whose area is 3200 sq cm.

5. Find the areas of circles whose circumferences are: (*a*) 25·4 in.; (*b*) 68·4 cm; (*c*) 124 ft.

6. A wall containing a circular window 3 ft in diameter and a door 6 ft 6 in. by 3 ft is 14 ft long and 11 ft high. Find in sq feet the area to be plastered ($\pi = \frac{22}{7}$). (U.L.C.I.)

7. The radius of a circle is 5 in. A chord is drawn at a distance of 3 in. from the centre. Find the areas of the two segments into which the circle is divided.

8. A circular plate of metal has a diameter of $1\frac{1}{2}$ ft. If twelve circular holes of diameter $\frac{1}{2}$ in. are drilled through it, what will the remainder weigh if 1 sq in. of the metal weighs 0·35 oz?

9. A piece of stonework is in the form of a rectangular slab 10 ft high and $4\frac{1}{2}$ ft wide, surmounted by a semi-circular slab. What would it cost to paint both sides of it at $2\frac{1}{2}d.$ per sq ft?

SECTION C

1. Find the following angles in radians subtended by the given arcs:

(*a*) arc = 11·4 in., radius = 2·4 in.;
(*b*) arc = 5·6 cm, radius = 2·2 cm.

2. Express the following angles in degrees, and minutes:

(*a*) 5 radians; (*b*) 0·234 radian; (*c*) 1·56 radians.

3. If ω = the angular velocity of a point in radians, and r = its distance from the centre of rotation, find the linear velocity of the point in the following cases:

(1) $\omega = 2{\cdot}5$ radians per sec, $r = 3{\cdot}64$ ft;
(2) $\omega = 4{\cdot}36$ radians per sec, $r = 4{\cdot}5$ ft;
(3) $\omega = 1{\cdot}48$ radians per sec, $r = 8{\cdot}2$ cm.

4. The linear velocity of a moving point P is 5·8 ft per sec. Its angular velocity with respect to a point Q is 4·8 radians per sec. What is the distance from P to Q?

5. A wheel is making 20 revolutions per minute. Find in radians the angle through which a spoke turns per sec. What is the linear velocity of a point on the spoke 2 ft 6 in. from the centre of the axle?

6. A water main is 20 in. in diameter, and is more than half full of water. The angle subtended at the centre by the horizontal surface of the water is $\frac{2\pi}{3}$ radians. Calculate:

(1) the length of the circumference that is wetted;
(2) the depth of the water. (U.E.I.)

7. A belt passing over a pulley 10 in. in diameter has 11 in. in contact with the pulley.

Find (1) in radians, (2) in degrees, the angle of the lap of the belt on the pulley.

(The angle of lap is the angle which the part of the belt in contact with the pulley subtends at the centre of the pulley.) (U.E.I.)

8. A circular arc is 12 ft 10 in. long, the radius of the arc is 7 yd. What is the angle subtended by the arc at the centre of the circle, in radians and degrees?

What length of arc would subtend the angle of 70° in the same circle? (U.L.C.I.)

9. Explain exactly the statement that a "radian" is the unit employed in the circular measure of an angle.

In a textile spinning-machine a reciprocating arm swings forward and backward through an angle of 88°. The forward motion takes 2·5 sec, the backward motion 11·5 sec. Find the average number of radians per sec, during the forward and backward swings. (U.L.C.I.)

MISCELLANEOUS

1. A keyway (a rectangular groove) is cut into a 2 in. dia shaft to the depth indicated in Fig. 101. What is the depth of bearing b of the key in the shaft?

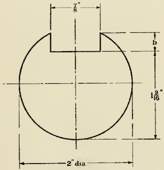


FIG. 101.

2. Find all the roots of the following equations which lie between 0° and 360°:

(a) $\cos\theta = -0{\cdot}3473$;
(b) $\tan\theta = -3{\cdot}04$;
(c) $\sin(2\theta - 37°) = 0{\cdot}5673$. (Rugby.)

3. Find all the values of θ between 0° and 360° for which $\sin\theta = -\frac{1}{2}$. (Coventry.)

4. A circular arc 14 in. long subtends an angle of 25° at the centre of the circle of which it is a part. Find the radius of the circle. (Coventry.)

5. The base of the segment of a circle is 16 ft long and its height is 4 ft. Calculate:

(i) the radius of the circle;
(ii) the length of the arc;
(iii) the area of the segment. (Coventry.)

6. Find the area of a circular sector of radius 7·5 in. and angle 25°, without using tables of degrees to radians. (Cannock.)

7. (a) Through what angle does each hand of a clock turn between one o'clock and half-past two?

(b) What is the time if the minute hand of a clock has turned through 75° since two o'clock? (Handsworth.)

8. At three o'clock the two hands of a clock are at right angles to one another. What is the shortest time which can elapse before the angle between them is again 90°? (Handsworth.)

9. (a) A line AB of length 6 ft revolves at 50 revolutions per minute about a perpendicular axis through A. Calculate:

(i) the speed of B in ft per sec;
(ii) the area swept out by the line in 0·1 sec.

(b) Find the rate, in feet per second, at which water is flowing through a pipe of 2 in. diameter if it delivers 3500 gal per hr. (1 cu ft of water = 6·24 gal.) (Surrey County Council.)

10. (i) Express in degrees angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, 2π, and 1·34 radians.

(ii) Express in radians an angle of 36° 45′.

(iii) A channel section is in the form of a segment of a circle of radius 8 in. The width of the channel is 6 in. Find the cross-sectional area of the channel.

(Nuneaton.)

11. A round bar of diameter d is machined to have a flat of width w, by removing metal to a depth h.

(a) Show that $w = 2\sqrt{dh - h^2}$.

(b) Transpose this formula into a form convenient for the calculation of d.

(c) If machining to a depth of 0·134 in. produces a flat of width 1 in. calculate the diameter of the bar. (Nuneaton.)

12. A locomotive travels at 40 m.p.h., and its driving-wheels are then turning at 160 revolutions per minute. Find the diameter of the driving-wheels.

A coil is wound on a cylindrical former 2·75 in. diameter, and consists of a single layer of 84 turns. Find the length of the wire used. (Worcester.)

CHAPTER 12

MENSURATION OF REGULAR SOLIDS

(see also earlier note on p. 40)

1. Units of Volume

The units employed in the measurement of volume are derived from those used in the measurement of length.

The Volume Unit is a cube whose edge is a unit of length.

Thus a cubic inch is a cube each edge of which is an inch in length.

A cubic centimetre (cc) is a cube each edge of which is a centimetre in length.

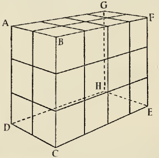


FIG. 102

2. Volume of a Square Prism

Suppose a number of cubes each having a volume of 1 cu in. to be arranged together as shown in Fig. 102.

The complete solid formed in this way is called a rectangular prism.

We notice that

(1) There are *three* layers of cubes.

(2) Each layer consists of *two* rows of cubes with *four* in each row.

Clearly there will be

$$3 \times 2 \times 4 \text{ cubes altogether}$$

—that is the volume $= 3 \times 2 \times 4$
$= 24$ cu in.

Treating this more generally,

Let BF contain l units of length.
,, AB contain b units of length.
,, BC contain h units of length.

There will then be h layers, and each layer will consist of b rows, with l cubes in each row. Hence the number of cubes will be $l \times b \times h$.

Let $\quad V =$ the volume.
Then $\quad \mathbf{V = lbh}$.

It also follows that $\quad \boldsymbol{l = \frac{V}{bh}}$

$$\boldsymbol{b = \frac{V}{lh}}$$

$$\boldsymbol{h = \frac{V}{lb}}$$

The area of the end ABCD is $(2 \times 3) = 6$ sq in., and the plane in which it lies is at right angles to the length of the prism.

In other words, it represents the area of a cross-section at right angles to the length of the prism.

Hence the volume V = Area of rectangle ABCD $\times$ length
= **Area of cross-section $\times$ length**

Similarly the area of the base is equal to the area of a section at right angles to the height.

Then $\quad$ V = Area of rectangle DCEH $\times$ height
= Area of base $\times$ height

We can generalise this as follows:

Let A $\equiv$ the area of a cross-section of the prism,
$h \equiv$ the dimensions at right angles to this section.

Then volume $\quad V = Ah$

From this, area of cross-section $\mathbf{A} = \dfrac{\mathbf{V}}{\boldsymbol{h}}$

and $\quad \boldsymbol{h} = \dfrac{\mathbf{V}}{\mathbf{A}}$

The Volume of a Cube

This is a special case of a rectangular prism in which $l = b = h$, so that if the edge of a cube be x in., its volume

$$V = x \times x \times x$$
$$= x^3 \text{ cu in.}$$

3. Volume of Any Prism

Fig. 103 represents a rectangular prism which is divided into two equal triangular prisms by the plane DBFH.

These triangular prisms stand on equal bases GFH and EFH.

The volume of either prism

$= \frac{1}{2}$ (Area of base EFGH $\times$ height)
= Area of triangular base $\times$ height

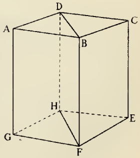


FIG. 103.

Let $\quad A \equiv$ area of base of triangular prism
$\qquad h \equiv$ its height
Then $\quad \mathbf{V = A}h$

The bases of these prisms are **right-angled** triangles. It is easy, however, to imagine other triangular prisms which have the same height as those shown in the figure, but whose bases are not *right-angled* triangles.

If the bases of such prisms have the same area as those shown in the figure, the **volumes** of the prisms will be the same.

It therefore follows that the volume of any triangular prism is obtained from the formula

$$V = Ah$$

Since any rectilineal figure can be built up from a number of triangles, we can extend this rule to all prisms, whatever their bases may be.

Hence for all prisms, if

$\qquad A \equiv$ Area of cross-section
$\qquad h \equiv$ its height

Then $\qquad \mathbf{V = A}h$

This is known as the **Prism Law.**

Example 1. *The external dimensions of a closed box are as follows: Length = 2 ft 3 in., width = 1 ft 2 in., height = 10 in. What is the minimum volume of wood required if it is $\frac{3}{8}$ in. thick?*

Since the wood is $\frac{3}{8}$ in. thick, the internal dimensions will be $26\frac{1}{4}$ in., $13\frac{1}{4}$ in. and $9\frac{1}{4}$ in.

Now, the volume of the wood will be the difference between the external and internal volumes of the box.

From the Prism Law,

$\qquad$ external volume $= (27 \times 14 \times 10)$ cu in.
$\qquad$ internal volume $= (26\frac{1}{4} \times 13\frac{1}{4} \times 9\frac{1}{4})$ cu in.

Hence, volume of wood

$$= (27 \times 14 \times 10) - (26\tfrac{1}{4} \times 13\tfrac{1}{4} \times 9\tfrac{1}{4}) \text{ cu in.}$$
$$= 3780 - 3217\tfrac{17}{64}$$
$$= 562\tfrac{47}{64} \text{ cu in.}$$
$$= 563 \text{ cu in. to the nearest cu in.}$$

Example 2. *The figure* ABCD (Fig. 104) *represents the cross-section of a trench* 40 *ft long. Find the weight in tons of the material removed if* 1 *cu ft weighs* 176 *lb.*

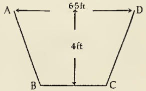


FIG. 104.

The trench forms a prism whose cross-section is the trapezium ABCD and whose length is 40 ft.

Let $\quad A \equiv$ the area of the trapezium

Then $\quad A = \left(\dfrac{6\cdot5 + 3\cdot5}{2}\right) 4$ sq ft.
$\qquad = 20$ sq ft.

Let $\quad V \equiv$ the volume of material removed
Now $\quad V = Ah$ where $h \equiv$ length of trench $= 5 \times 40$.
$\therefore \quad$ Weight of material $= 20 \times 40 \times 176$ lb
$$= \frac{20 \times 40 \times 176}{2240} \text{ tons}$$
$$= 62\cdot85 \text{ tons approx.}$$

Example 3. *The diagram* (Fig. 105) *represents the cross-section of an angle iron in which* FE = CB = $\frac{3}{8}$ *in.,* AB = 2·5 *in. and* AF = 3·3 *in. Its length is* 18 *ft. Find its weight, if* 1 *cu in. weighs* 0·28 *lb.*

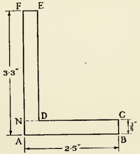


Fig. 105.

Area of cross-section

= Area of rectangle ABCN + rectangle NFED
= (2·5 × 0·375) + (2·925 × 0·375)
= 5·425 × 0·375 sq in.

The angle iron forms a prism so that

$$V = Ah$$

and $h = 18$ ft $= 216$ in.

$\therefore$ V = 5·425 × 0·375 × 216 cu in.

Hence weight = 5·425 × 0·375 × 216 × 0·28 lb
= 123 lb approx.

THE CYLINDER

4. Surface

A cylinder is a regular solid which we can look upon as being formed by the rotation of a rectangle about *one* of its sides.

Thus, as in the figure (Fig. 106), let the rectangle ABCD rotate about AB.

Then AB is the axis of the cylinder.

Any point P in DC will, at the same time, form a circle, and this circle will be a section of the cylinder at right angles to the axis AB.

Let $r \equiv$ the radius of cross-section of a cylinder.
$h \equiv$ its height.

Now, the area of the **curved** surface of a cylinder is clearly equal to that of a rectangle whose length is equal to the circumference of any cross-section, and whose height is equal to the height of the cylinder.

This can be very easily verified by first wrapping a piece of smooth paper round the cylinder so that it is exactly covered, and then opening out the paper on the flat.

Hence area of curved surface

$$= 2\pi r \times h$$
$$= 2\pi rh$$

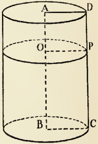


Fig. 106.

Also area of each end of the cylinder $= \pi r^2$.

Let $S \equiv$ the total surface.

Then $S = 2\pi rh + 2\pi r^2$

or $\mathbf{S = 2\pi r(h + r)}$

See also No. 24 Miscellaneous Exercises, Chapter 2.

5. Volume

As already stated, the cross-section of a cylinder at right angles to its axis is a circle, and since a circle can be regarded in the limit as a regular polygon with an infinite number of sides (see p. 258, § 3), we can treat the cylinder as a prism, and its volume will thus be found from the Prism Law.

Let $A \equiv$ the area of its base or cross-section
$h \equiv$ its height

Then (see Fig. 107) $V = Ah$
But $A = \pi r^2$, where $r \equiv$ the radius of base.
Then $\mathbf{V = \pi r^2 h}$

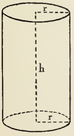


Fig. 107.

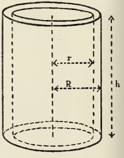


Fig. 108.

6. Volume of a Hollow Cylinder

The volume of the material contained in a hollow cylinder can be expressed as the difference between the volumes of two cylinders.

Let R and r (Fig. 108) be the external and internal radii of the hollow cylinder, and h its height.

Let $V \equiv$ the volume of the material.

Then
$$\mathbf{V} = \pi\mathbf{R}^2\boldsymbol{h} - \pi\boldsymbol{r}^2\boldsymbol{h}$$
$$= \pi\boldsymbol{h}(\mathbf{R}^2 - \boldsymbol{r}^2)$$
$$= \pi\boldsymbol{h}(\mathbf{R} + \boldsymbol{r})(\mathbf{R} - \boldsymbol{r})$$

Example 1. *A cylindrical metal tank is required to hold 25 gal with a diameter not exceeding 2 ft.*

Find:

(1) *Its height.*
(2) *Minimum amount of sheet metal required.*

(1) Since 1 cu ft $= 6\frac{1}{4}$ gal

the volume $V = \dfrac{25}{6\frac{1}{4}} = 4$ cu ft.

But $V = \pi r^2 h$, where $h \equiv$ the height, $r \equiv$ the radius.

Then $4 = 3{\cdot}142 \times 1^2 . h$

$\therefore \quad h = \dfrac{4}{3{\cdot}142} = 1{\cdot}273$

(2) Let $S \equiv$ the total external surface of the tank.

Then
$$S = 2\pi r(h + r)$$
$$= 2 \times 3{\cdot}142 \times 1 \times 2{\cdot}273$$
$$= 14{\cdot}28 \text{ sq ft.}$$

Example 2. *A cylindrical pipe is 8 ft long, 5 in. internal diameter and $\frac{1}{2}$ in. thick. Find its weight if the material is 7·8 times as dense as water.*

Taking 1 cu ft of water to weigh 62·5 lb, 1 cu ft of the material of the pipe will weigh (62·5 × 7·8) lb.

Since the pipe is really a *hollow cylinder*, the volume of the material V is obtained from

$$V = \pi h(R + r)(R - r) \text{ as above;}$$

then $V = \frac{22}{7} \times 96 \times 5\frac{1}{2} \times \frac{1}{2}$ cu in.

$$= \frac{22}{7} \times \frac{96 \times 5\frac{1}{2} \times \frac{1}{2}}{1728} \text{ cu ft.}$$

$\therefore$ Its weight

$$= \frac{22}{7} \times 96 \times \frac{11}{2} \times \frac{1}{2} \times \frac{62 \cdot 5 \times 7 \cdot 8}{1728} \text{ lb}$$

$$= 234 \text{ lb.}$$

Volume of a Pyramid

7. Fig. 109 represents a cube standing on its base DBCE. The diagonals of the cube PB, ND, CQ and ME intersect at the centre A.

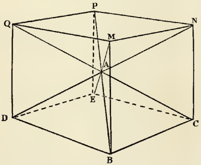


Fig. 109.

The lower halves of these diagonals—namely, AB, AC, AE and AD—form the slant edges of a pyramid standing on the base BCED, and the height of this pyramid is half that of the cube.

Each face of the cube forms the base of a similar pyramid with its apex at A, and having the same height and the same

volume as ABCED, so that the cube can be considered as being built up of six pyramids equal in volume.

Let a = the edge of the cube.

Then its volume = a^3.

$\therefore$ Volume of one pyramid $= \frac{1}{6}a^3$

and its height $= \frac{1}{2}a$

Now $\frac{1}{6}a^3 = \frac{1}{3} \times a^2 \times \frac{1}{2}a$

that is **Volume of pyramid** = $\frac{1}{3}$ × **area of base** × **its height.**

But **Area of base** × **height** = **Volume of the corresponding prism.**

Hence we can say *that the volume of a pyramid is one-third of the corresponding prism having the same base and height.*

that is $\mathbf{V = \frac{1}{3}A}h$,

where A is the area of the base.

Though we have only dealt here with a pyramid on a square base, the rule is applicable to all pyramids.

Example. *Find* (a) *the volume of a pyramid whose base is a regular hexagon of* 1 *in. side and whose height is* 5 *in.*
(b) *Find also the area of its sloping surfaces.*

(*a*) **Volume of pyramid**

If A = the area of the hexagon ABCDEF (Fig. 110)
and h = the height PQ
then $V = \frac{1}{3}Ah$
Now A = 6 times equilateral $\triangle$QBC.

Draw QN perpendicular to BC. Then QN bisects BC.

Area of $\triangle QBC = \frac{1}{2}BC \times QN$

$$= \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2} \text{ sq in.}$$

$$= \frac{\sqrt{3}}{4} \text{ sq in.}$$

Then **area of hexagon** $= \frac{6\sqrt{3}}{4}$ sq in.

$\therefore$ **Volume of pyramid** $= \frac{1}{3} \times \frac{6\sqrt{3}}{4} \times 5$ cu in.

$= 4{\cdot}33$ cu in.

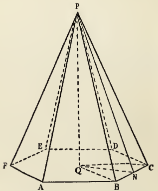


Fig. 110.

(*b*) The sloping surface consists of six equal triangles. Consider the $\triangle PBC$, which is isosceles.

Then PN drawn $\perp$ to BC bisects BC and meets the perpendicular from Q to BC.

$$\textit{Area of } \triangle PBC = \tfrac{1}{2} \cdot BC \cdot PN$$

Since $\triangle PQN$ is right-angled at Q

$$\begin{aligned} PN^2 &= PQ^2 + QN^2 \\ &= 5^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 25 + \tfrac{3}{4} \\ &= 25{\cdot}75 \end{aligned}$$

$$\therefore \quad PN = \sqrt{25{\cdot}75}$$

Hence *area of* $\triangle PBC = \frac{1}{2} \times 1 \times \sqrt{25{\cdot}75}$

$\therefore$ Area of sloping surface $= \frac{1}{2} \times 1 \times \sqrt{25{\cdot}75} \times 6$

$= 15{\cdot}22$ sq in.

Note.—(1) The line PN drawn perpendicular to BC is called the *slant* height of the pyramid. (2) PB and PC are called *slant* edges.

The Cone

8. Surface

In the last worked example we dealt with a pyramid on a **hexagonal** base.

It is easy to imagine a pyramid on a base having a very much larger number of sides, and to see that as the number of sides increases, the pyramid approaches the form of a cone.

Hence, ultimately, as the number of sides increases indefinitely we obtain a cone as a special case of a pyramid.

If a right-angled triangle PQS rotates about PQ as an axis, the solid traced out by a complete rotation will be the cone PRMS (Fig. 111).

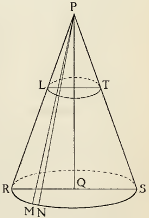


Fig. 111.

Any point T on PS will trace out a circle, and this circle will be at right angles to the axis PQ.

Take two points M and N very close together on the circumference of the base and join to P.

Then PMN is approximately a triangle, and the closer M and N approach one another, the closer is the approximation. At the same time, the perpendicular from P to M approximates very closely to the slant height PM. We can therefore look upon the curved surface as being built up of an infinite number of small triangles of height PM.

Now, area of the triangle

$$= \tfrac{1}{2} \times \text{slant height of cone} \times \text{MN}$$

$\therefore$ Sum of areas of all such triangles is

$\tfrac{1}{2} \times$ slant height of cone $\times$ (sum of bases such as MN)

and the sum of these bases such as MN ultimately approximates to the circumference of the base of the cone.

$\therefore$ Area of curved surface of a cone
$= \tfrac{1}{2}$ circumference of base $\times$ slant height.

Let r = radius of base,
l = slant height.

Then circumference of base $= 2\pi r$

$\therefore$ **Area of curved surface** $= \tfrac{1}{2} \times 2\pi r \times l$
$= \boldsymbol{\pi r l}$

Let $h \equiv$ the height of the cone.

Then from the figure, PQS being a right-angled triangle

$$l^2 = h^2 + r^2$$

that is $\quad l = \sqrt{h^2 + r^2}$

$\therefore$ Area of curved surface $= \pi r\sqrt{h^2 + r^2}$

If we take the base into consideration as well,

Total Surface $= \pi r l + \pi r^2$
$= \boldsymbol{\pi r(l + r)}$

It should be noted that the two cones PLT and PRS have the same vertical angle, and therefore they are similar.

Example. *A tent is in the form of a cylinder surmounted by a cone. Find the total area of canvas required if the height of the tent is* 18 *ft, and height of cylindrical portion is* 12 *ft, with a diameter of* 26 *ft.*

(*a*) **Slant height of conical portion** $l = \sqrt{r^2 + h^2}$
$= \sqrt{6^2 + 13^2}$
$= 14{\cdot}32$ ft.

$\therefore$ Since Area $= \pi r l$
Area $= \frac{22}{7} \times 13 \times 14{\cdot}32$ sq ft
$= 585$ sq ft.

(*b*) **Cylindrical portion of tent**

If $h \equiv$ the height and $r \equiv$ radius of base.

Area $= 2\pi r \times h$
$= 2 \times \frac{22}{7} \times 13 \times 12$
$= 980{\cdot}6$ sq ft.

$\therefore$ Total canvas required $= 585 + 980{\cdot}6$ sq ft
$= 1565{\cdot}6$ sq ft.

9. Volume of a Cone

Since, as has already been shown, we can consider the cone as a special case of a pyramid, the formula $V = \frac{1}{3}Ah$ can be applied in determining its volume.

Now A, the area of the base $= \pi r^2$

$\therefore$ **Volume of a cone** $= \boldsymbol{\tfrac{1}{3}\pi r^2 h}$.

Hence the *volume of a cone is one-third the volume of a cylinder of the same height and base.*

Example. *A pyramid stands on a square base of side* 8 *in. What is the radius of the base of a cone having the same volume and height?*

For the pyramid $\quad V = \frac{1}{3}Ah$

For the cone $\quad V = \frac{1}{3}\pi r^2 h$

Then $\frac{1}{3}\pi r^2 h = \frac{1}{3}Ah$

$\therefore \quad \pi r^2 = A$

that is $\frac{22}{7}r^2 = 64$

$\therefore \quad r = \sqrt{64 \times \frac{7}{22}}$

$= 4{\cdot}51$ in. correct to 0·01 in.

Volumes of Similar Solids

10. We have seen in a previous chapter that the *areas* of *similar figures* are proportional to the squares of the corresponding linear dimensions.

A similar rule applies to similar solids with regard to their *volumes*.

It is stated as follows:

The Volumes of similar solids are proportional to the cubes of the corresponding linear dimensions.

Illustrations

(1) The rule obviously applies to two cubes, for if x_1 and x_2 are the edges of two cubes, and V_1 and V_2 the corresponding volumes

$$V_1 = x_1^3 \text{ and } V_2 = x_2^3$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{x_1^3}{x_2^3}$$

Let ABC and DEF be two similar cones (Fig. 112) whose heights are h_1 and h_2, and the radii of whose bases are r_1 and r_2.

Since $\angle BAC = \angle EDF$

$\angle P_1AC = \angle P_2DF$

$\therefore$ The $\triangle$s P_1AC, P_2DF are similar.

Hence $$\frac{h_1}{h_2} = \frac{r_1}{r_2}$$

$$\therefore \quad \frac{h_1^3}{h_2^3} = \frac{r_1^3}{r_2^3}$$

Then if V_1 and V_2 are the respective volumes

$$\frac{V_1}{V_2} = \frac{\frac{1}{3}\pi r_1^2 h_1}{\frac{1}{3}\pi r_2^2 h_2} = \frac{r_1^2 h_1}{r_2^2 h_2}$$

$$= \frac{r_1^2}{r_2^2} \times \frac{h_1}{h_2} = \frac{r_1^3}{r_2^3} = \frac{h_1^3}{h_2^3}$$

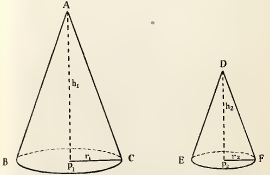


Fig. 112.

ABCDE is a pyramid on a square base and GHKF is a section parallel to the base (Fig. 113).

Then AGHKF is a similar pyramid. Let h_1 and h_2 be their respective heights and V_1 and V_2 their volumes.

The triangles AOH and APB are similar.

Then $$\frac{h_1}{h_2} = \frac{AB}{AH}$$

Also the triangles AGH and ACB are similar.

Then $$\frac{AB}{AH} = \frac{BC}{HG}$$

Hence from the rule with regard to volumes

$$\frac{V_1}{V_2} = \frac{h_1^3}{h_2^3} = \frac{AB^3}{AH^3} = \frac{BC^3}{HG^3}$$

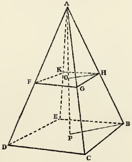


FIG. 113.

11. Weights of Similar Solids

The weight of a body is proportional to its volume.

Hence if V_1 and V_2 are the volumes of two bodies having the same density and W_1 and W_2 their weights

$$\frac{W_1}{W_2} = \frac{V_1}{V_2}$$

$\therefore$ It follows that the *weights of similar solids of the same density are proportional to the cubes of the corresponding linear dimensions.*

Example. *The volume of a cone of height* 12·8 *in. is* 180 *cu in. Find the height of a similar cone whose volume is* 70 *cu in.*

From the rule given above, if h_1 and h_2 represent the heights and V_1 and V_2 the corresponding volumes.

$$\frac{V_1}{V_2} = \frac{h_1^3}{h_2^3} \text{ (see Fig. 112)}$$

$$\therefore \quad \frac{180}{70} = \frac{12{\cdot}8^3}{h_2^3}$$

that is, $$h_2^3 = \frac{70 \times 12{\cdot}8^3}{180}$$

and $$h_2 = \sqrt[3]{\frac{70 \times 12{\cdot}8^3}{180}}$$

Taking logs of both sides

$$\log h_2 = \tfrac{1}{3}(\log 70 + 3 \log 12{\cdot}8 - \log 180)$$
$$= 0{\cdot}9704$$
$$\therefore \quad h_2 = 9{\cdot}342 \text{ in.}$$

THE SPHERE

12. A sphere is a solid such that every point on its surface is equidistant from a fixed point within it, which is called the **centre.**

We can consider a sphere as being formed by the rotation of a semi-circle about a diameter.

NOTE.—(1) Any section of a sphere is a circle.
(2) All sections which contain the centre are equal in area. These are sometimes called the *great circles*.

Though for practical purposes it is necessary that the student should be acquainted with the formulæ by which the **area of the surface** and **the volume of a sphere** can be calculated, the proofs will not be given, as they involve a knowledge of mathematics which is beyond the scope of this book.

13. Area of the Surface of a Sphere

Let $R \equiv$ the radius of sphere

$A \equiv$ the area

Then $\mathbf{A = 4\pi R^2}$

14. Volume of a Sphere

Let $R \equiv$ the radius of a sphere

$V \equiv$ its volume

Then $\mathbf{V = \frac{4}{3}\pi R^3}$

Let $D \equiv$ the diameter $= 2R$

Then
$$\begin{aligned} V &= \tfrac{4}{3}\pi \cdot \frac{D^3}{8} \\ &= \tfrac{1}{6}\pi \cdot D^3 \\ &= 0{\cdot}5236D^3 \end{aligned}$$

Example 1. *Find the weight of a hemispherical bowl of copper whose external and internal radii are* 10 *cm and* 9 *cm respectively. Take the density of copper as* 8·9 *gm per cc.*

Let R_1 and R_2 be the radii.

$$\begin{aligned} \text{The volume of material} &= \tfrac{1}{2}(\tfrac{4}{3}\pi R_1^3 - \tfrac{4}{3}\pi R_2^3) \\ &= \tfrac{2}{3}\pi(R_1^3 - R_2^3) \\ &= \tfrac{2}{3} \times \tfrac{22}{7}(1000 - 729) \text{ cc.} \\ &= \tfrac{2}{3} \times \tfrac{22}{7} \times 271 \text{ cc.} \end{aligned}$$

$$\begin{aligned} \text{Hence weight of material} &= \tfrac{2}{3} \times \tfrac{22}{7} \times 271 \times 8{\cdot}9 \text{ gm} \\ &= 5053{\cdot}5 \text{ gm} \\ &= 5{\cdot}054 \text{ kg approx.} \end{aligned}$$

Example 2. *The cost of plating a metal sphere at* 5s. 9d. *per sq ft is* £2 10s. 0d. *Find its diameter.*

$$\begin{aligned} \text{Area plated} &= \frac{50}{5\frac{3}{4}} \text{ sq ft} \\ &= \frac{50 \times 4}{23} \text{ sq ft.} \end{aligned}$$

Hence if R be the radius of the sphere

$$4\pi R^2 = \frac{50 \times 4}{23}$$

Then
$$\begin{aligned} R^2 &= \frac{50 \times 4}{4 \times 3{\cdot}142 \times 23} \\ \therefore\ R &= \sqrt{\frac{50}{3{\cdot}142 \times 23}} \text{ ft.} \\ &= 12\sqrt{\frac{50}{3{\cdot}142 \times 23}} \text{ in.} \\ &= 10{\cdot}00 \text{ in. (correct to } 0{\cdot}01) \end{aligned}$$

$\therefore$ **Diameter** $= 20$ in. (approx.)

15. Use of Tabulated Matter

It is essential that the student should use the *rules* of mensuration with confidence. It is better that the formulæ should serve merely as reminders of the *rules* than as expressions to be evaluated by unintelligent substitution. It is well known among teachers that an answer obtained by blind substitution is commonly stated in wrong units.

One advantage of a clear understanding of the rules of mensuration is that tabulated information can be safely introduced in order to shorten the calculation. Engineers commonly simplify their computations a good deal by doing this.

Example. Thus the 6 in. × 3 in. rolled steel joist of the example on p. 39 is given by reference-book tables as weighing 12·0 lb per ft run correct to three significant figures. The " web " of this joist is 0·25 in. thick.

By what percentage would a beam made from this rolled section be lightened if holes 3 *in. dia were cut in the web, spaced* 5 *in. centre to centre?*

The area of one 3 in. dia hole is

$$\frac{\pi}{4} \times 3^2 \text{ sq in.}$$

The volume of metal removed is obtained by multiplying this area by the thickness 0·25 in. The weight of metal removed is in turn obtained by multiplying the volume by the density of the rolled steel, which is 0·284 lb per cu in.

Weight of metal removed per hole is

$$\frac{\pi}{4} \times 3^2 \times 0{\cdot}25 \times 0{\cdot}284 \text{ lb}$$
$$= 0{\cdot}503 \text{ lb}.$$

Now there is one hole for each 5 in. run of the joist. At 12·0 lb per ft, a run of 5 in. weighs 5 lb.

So the percentage saving in weight through boring the holes is

$$\frac{0{\cdot}503 \text{ lb}}{5 \text{ lb}} \times 100$$

Cancelling by 1 lb, we arrive at the percentage figure

10·06,
or 10·1 to three significant figures.

Notice that the unit of weight, 1 lb, cancels. The percentage would have worked out the same if the weights had been given, say, in kilograms.

EXERCISE XII

At the discretion of the teacher students may proceed directly to Section D, Miscellaneous Exercises.

Section A

Prisms

1. A bar of metal is 3 ft 6 in. long, 3 in. wide and $1\frac{1}{2}$ in. thick. Find its volume, and weight, if 1 cu in. weighs 5·4 oz.

2. If an inch = 2·54 cm, express a cubic inch in cc.

3. A rectangular room is 50 ft long, 30 ft wide and 12 ft high. If it is occupied by 45 persons, how many cubic feet of air are available for each person?

4. A rectangular tank is 4 ft long and 3 ft wide, and contains a certain amount of water. If on dropping a solid into it the water rises $1\frac{1}{2}$ in, what is the volume of the solid?

5. If in the previous question the tank is 2 ft deep, find its capacity in gallons if 1 gal = 277·3 cu in.

6. The cross-section of a rectangular beam is 150 sq in. If its length is 16 ft, find its weight if 1 cu ft weighs 36 lb.

7. A cubic foot of lead is hammered out in order to make a square sheet $\frac{3}{8}$ in. thick. What is the area of the square?

8. The concrete foundation for a wall is 1 ft 4 in. thick and 3 ft wide. Calculate the weight in tons of the concrete required for a foundation 40 ft long if 1 cu ft weighs 133 lb.

9. A tank is required to contain 250 gal of water. If the length is 3 ft, and the width 2 ft, what must be its depth? (Take 1 cu ft = $6\frac{1}{4}$ gal.)

10. The internal dimensions of a wooden box without a lid are: length = $3\frac{1}{2}$ ft, width = 3 ft, depth = 2 ft.

If the wood be $\frac{1}{2}$ in. thick, calculate the volume of wood required.

11. A prism has an equilateral triangle of 1-in. side as its base, and its length is 15 in. What is its volume?

12. The internal cross-section of a feeding-trough is 2·4 sq ft and its length is 12 ft. What is its capacity in gallons?

Section B

Cylinders

1. Find the total surface area of the cylinders in which

(*a*) Height of cylinder = 12·5 in., radius of base = 2·4 in.

(*b*) Height of cylinder = 3·4 ft, radius of base = 0·6 ft.

2. What would it cost to paint the curved surface of four cylindrical pillars 24 ft high, and whose radius of cross-section is 9 in. at $2\frac{1}{2}d.$ per sq ft?

3. A garden roller is $2\frac{1}{2}$ ft long and has a diameter of 21 in. What area of ground would be covered by it in 140 complete revolutions? ($\pi = \frac{22}{7}$.)

4. If a system of heating by hot water is composed of 980 ft of 4-in. pipes (external diameter), find in square feet the surface area of piping giving out heat. ($\pi = \frac{22}{7}$.) (U.L.C.I.)

5. A cylindrical tank closed at both ends is to be made of sheet metal. The diameter of the base is to be 3 ft 6 in., and the height 5 ft 3 in. Find the surface area of the sheet metal required. (U.L.C.I.)

6. Find the volumes of the cylinders with the following dimensions:

(*a*) Diameter of base 3 in., height 6 in.
(*b*) Diameter of base 15 cm, height 34 cm.

7. If R and r are the external and internal radii of a hollow cylinder, find the volume of material in each of the following cases:

(*a*) $R = 1{\cdot}25$ ft, $r = 1{\cdot}1$ ft and $h = 16$ ft;
(*b*) $R = 8$ in., $r = 6{\cdot}5$ in. and $l = 3$ ft 6 in.

8. A flywheel has a diameter of 2 ft and its thickness is 3·6 in. Find its weight if a cubic foot of the metal of which it is made weighs 487·5 lb.

9. The volume of a cylinder is 220 cc, and the radius of cross-section is 3 cm; find its height.

10. What length of wire of diameter 0·6 mm can be made from 630 cc of copper? ($\pi = \frac{22}{7}$.)

11. Find the weight of 24 ft of steel shafting if the diameter is 8 in. and 1 cu in. weighs 0·28 lb.

12. (*a*) A circular metal washer has a square piece cut out. If the diameter of the washer is 2R, the thickness

t, and the side of the square l, express the volume V by means of a formula.

(*b*) If $R = 5{\cdot}02$ in., $t = 0{\cdot}19$ in., $l = 6{\cdot}01$ in., find V. ($\pi = 3{\cdot}14$.) (U.E.I.)

Section C

Pyramids, Cones and Spheres

1. A pyramid 12 in. high stands on a square base of 6 in. side. Find (*a*) its volume, (*b*) its total surface area.

2. Find the volume of a pyramid which stands on a hexagonal base of 1·5 cm side, and has a height of 8 cm.

3. Find the volumes of the cones of the given dimensions

(*a*) Radius of base 4·5 in., height 9 in.
(*b*) Radius of base 1·8 ft, height 12 ft.

4. Find also the total surface area of the cones in Question 3 (*a*), (*b*).

5. A conical heap of earth has a slant height of 10 ft, and the circumference of the base is 32 ft. What is its volume?

6. The vertical angle of a cone is 60°, and the radius of the base is 1·4 in.

Find (*a*) its volume, (*b*) its curved surface.

7. The area of the curved surface of a cone is 22·4 sq in. and the slant height is 8 in. Find the area of the base of the cone.

8. The curved surface of a cone is 20·48 sq in. What is the curved surface of a similar cone whose height is 1·4 times that of the first?

9. The heights of two similar cones are in the ratio of 2 : 3. If the volume of the smaller is 15 cu in., what is the volume of the larger?

10. A pyramid of metal standing on a square base of 6 in. side weighs 100 lb. What would be the weight of a similar pyramid the edge of whose base is $4\frac{1}{2}$ in.?

11. Find the areas of the surfaces of the spheres whose radii are (*a*) 2·4 cm, (*b*) 5·6 in.

12. Find the volumes of the spheres whose radii are (*a*) 1·6 cm, (*b*) 4·2 in., (*c*) 2·5 ft.

13. What would it cost to electro-plate a metal sphere of 2 ft diameter at 5*s.* 6*d.* per square foot?

SECTION D

Miscellaneous

1. The following information relating to steel bars is taken from an engineers' reference book:

Section	lb per ft run
1 in. dia	2·68
2 in. ,,	10·7
3 in. ,,	24·1
4 in. ,,	42·8
1 in. square	3·41
2 in. × 1 in. flat	6·82

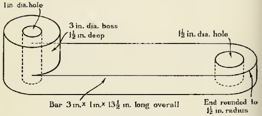


FIG. 114

Use this information to calculate as simply as possible the weight of parts made to the sketches Figs. 114 and 115. (Based on C.G.L.I.)

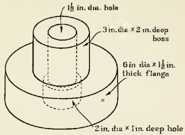


FIG. 115.

2. A steel rivet is in the shape of a cylinder surmounted by a hemisphere. The diameter of the cylinder is $\frac{1}{2}$ in. and of the head 1 in.; the greatest length of the rivet is 2 in. Find the weight of 100 rivets if steel weighs 0·28 lb per cu in. (Rugby.)

3. What would be the diameter of a cylindrical petrol tank 6 ft long to hold 250 gal? (1 cu ft = $6\frac{1}{4}$ gal.) (Rugby.)

4. (*a*) What area of canvas will be required for a conical tent 12 ft high and 10 ft base diameter?

(*b*) It is estimated that a spherical observation balloon will require to be 18 ft in diameter. How many cubic feet of gas will it contain when fully inflated? What is the total area of fabric needed to cover it with a double layer? (Rugby.)

5. The rim of a cast-iron flywheel is 6 in. wide and 6 in. thick and of outside diameter 8 ft. Calculate the weight of the rim if 1 cu in. of iron weighs 0·26 lb. (Burton upon Trent.)

6. (i) A solid lead cone, 12 in. high and of base radius 3 in., is melted and recast into two identical spheres. Find the radius of each sphere.

(ii) The diameter of a bicycle wheel is 28 in. If the wheel rolls through 5 revolutions in 2 sec, find the speed of the bicycle in miles per hour. ($\pi = \frac{22}{7}$.) (Sunderland.)

7. A container (steel-works) is in the form of a cylinder of height 10 ft and diameter 8 ft; its bottom end is a hemisphere of the same radius as the cylinder. Three-fifths of the total volume is filled with molten metal 1 cu ft of which weighs 450 lb. Find the weight in kilograms of the metal in the container assuming the measurements given refer to the internal dimensions of it. (Sunderland.)

8. Find the capacity of a bucket made in the shape of a frustum of a right circular cone, height $10\frac{1}{2}$ in., diameter of ends 11 in. and $5\frac{1}{2}$ in. (Take $\pi = \frac{22}{7}$ and 1 gal $= 277{\cdot}0$ cu in.) (Sunderland.)

9. A hollow rectangular block with closed ends has a length of L ft. The cross-section of the cavity is a square of side x in. The metal is t in. thick and weighs z lb/cu in. Determine the volume and weight of the block. (U.L.C.I.)

10. A cone 40 in. high has to be cut parallel to its base so that the resulting smaller cone is three-quarters of the weight of the original cone. What should be the height of the smaller cone? (U.L.C.I.)

11. A brass tube 9 ft long has an outside diameter 3 in. and inside diameter 2·8 in. Calculate the volume of brass in cubic inches.

If a cubic inch of brass weighs 0·3 lb, what is the weight of the tube? (U.L.C.I.)

12. Hemispherical bowls for plumbers' ladles are to be cast in iron, 75 at one pouring. What weight of melted metal will be needed if each bowl is 5 in. internal diameter and the metal is $\frac{3}{16}$ in. thick? (1 cu in. of iron weighs 0·26 lb.) (Nuneaton.)

13. The diagram shows the vertical cross-section of a metal bucket which is strengthened by means of a circular ring which is rigidly attached to the bucket at a vertical height of $1\frac{1}{2}$ ft above the base. Find:

(a) the height of the cone of which the bucket is a part;

(b) the circumference of the ring where it contacts the bucket;

(c) the angle of slant, θ, of the bucket to its base.

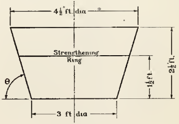


FIG 116

(Surrey County Council.)

14. A frustum of a cone has end diameters of 6 in. and 18 in. and a height of 8 in.

(i) Calculate the height of the cone of which the frustum might have formed the lower part.

(ii) Calculate the volume of the frustum by considering it as the difference between the volumes of two cones.

(iii) Calculate the area of the curved surface of the frustum by using a method similar to the one you used for calculating the volume.

(Nuneaton.)

15. A quadrant is cut out of a piece of circular metal of radius 11·3 cm, and the remainder is bent to form a cone. Find the base radius and height of the cone. (E.M.E.U.)

16. The diagram shows the vertical cross-section of a glass electric light shade which consists of a hemisphere surmounted by two cylinders. If the topmost cylinder has a circular aperture of diameter 4 in., find the surface area of glass in the shade.

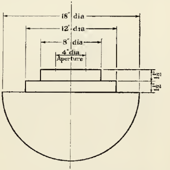


FIG. 117.

(Surrey County Council.)

17. Water flows in a $3\frac{1}{2}$-in. pipe at the rate of 10 ft per sec. How many cubic feet are delivered per hour?

(W.R. Yorks.)

18. (*a*) A rectangular tank 4 ft long, $3\frac{1}{2}$ ft wide and 4 ft deep is half full of water. A metal sphere of diameter 18 in. is placed in the tank. Calculate the new depth of the water.

(*b*) A pipe of 3 in. internal diameter is running full of water at 7 ft per sec. Calculate the discharge in gallons per minute. (1 cu ft $= 6\frac{1}{4}$ gal.) (Shrewsbury.)

19. (*a*) Write down an expression for finding the area of the curved surface of a cylinder, explaining the symbols used.

(*b*) Calculate the amount of cooling surface provided by the tubes of a surface condenser if there are 1000 such tubes, each 6 ft long and 1 in. outside diameter.

(Worcester.)

20. If the ratio of the weights of two spheres made of the same material is 27 : 8, find the ratio of:

(i) the radii;
(ii) the surface areas. (Coventry.)

21. The average speed of water flowing along a pipe is 3 ft per sec. What volume of water will pass through any particular section in $\frac{1}{4}$ min if the diameter of the pipe is 8 in.?

22. The cross-section of a wedge is an isosceles triangle whose sides are 8 in., 8 in. and 3 in.

If the width at right angles to this section be 9 in., find its weight, taking 1 cu in. to weigh 0·026 lb.

23. The interior cross-section of a water-trough is a semicircle of diameter 24 in.

If the length of the trough be 12 ft, how many gallons does it contain? (Take 1 cu ft $= 6\frac{1}{4}$ gal.)

24. A cast-iron dumb-bell consists of two spheres of $2\frac{1}{2}$ in. diameter connected by an iron cylinder 6 in. long and 1 in. diameter.

Find its weight if 1 cu in. weighs 0·26 lb.

25. A square metal plate of side L, thickness, t, has a circular hole in it of radius r.

(*a*) Give a formula for the volume of the metal.
(*b*) If $L = 10$ cm, $t = 0·67$ cm, $r = 4·4$ cm, what percentage of the metal was cut away when the hole was made?

26. Given that the weight of 1 cu in. of copper is 0·32 lb, calculate the weight of a copper tube of internal diameter 1 in., with wall thickness $\frac{1}{8}$ in., and of length 4 ft 6 in.

Find, by proportion, the weight of a similar tube of aluminium.

The weight of 1 cu in. of aluminium is 0·098 lb.

(U.E.I.)

27. A brass plate for a condenser is $\frac{3}{4}$ in. thick and is in the form of a rectangle 2 ft $11\frac{1}{2}$ in. long by $23\frac{3}{4}$ in. broad. Each corner is rounded off to a radius of 3 in. Sketch the plate and calculate its weight. (1 cu in. of brass weighs 0·3 lb.) (U.E.I.)

28. The diameter of a gas-engine cylinder is 6·5 in. and the stroke of the piston is 12 in. Calculate the stroke volume (V), *i.e.* the volume swept through by the piston in one stroke. If the clearance volume (c) is 30% of the stroke volume, determine the clearance volume.

Find also the compression ratio (r) from the formula

$$r = \frac{V + c}{c}$$

(U.L.C.I.)

29. Copper is 8·9 times as heavy as water. Find the weight in pounds of a copper wire 1000 ft long and 0·01 in. in diameter. Find also the weight of the same length of copper wire when the diameter is 0·1 in. (U.L.C.I.)

30. A cylindrical tank, open at the top, is made of sheet metal which weighs 1·8 lb per sq ft. The diameter of the tank is 2 ft 6 in., and its depth is 8 ft. Allowing 20% additional metal for joints and stiffening, find the weight of the tank (a) when empty, (b) when full of water. (1 cu ft of water weighs 62·3 lb.) (U.L.C.I.)

31. The areas of the surfaces of four spheres are to one another as $1 : 36 : 64 : 81$. Find the ratio of the volume of the largest sphere to the sum of the volumes of the other three. (N.C.T.E.C.)

32. A hollow closed cubical box is made of metal 1 in. thick. The length of each outside edge of the box is 2 ft. Find the weight of the box, given that 1 cu in. of the metal weighs 0·3 lb. (N.C.T.E.C.)

33. The breadth and height of a rectangular block are equal; the length is five times the breadth. Obtain a formula for the total surface area in terms of the height. If the total surface area is 198 sq in., calculate the height and also the volume. (N.C.T.E.C.)

34. A steel plate 1 in. thick is in the form of a portion of a circle bounded by two radii 4 ft 7 in. long, which include an angle of $54\frac{1}{2}°$. Calculate the weight of the plate. (1 cu in. of steel weighs 0·28 lb.) (U.E.I.)

35. A storage tank is in the form of a horizontal cylinder with hemispherical ends. Total overall length is 6 ft 6 in. and length of cylindrical portion is 4 ft. Calculate in gallons the quantity of liquid stored when the tank is half full. (1 cu ft = $6\frac{1}{4}$ gal.) (U.E.I.)

36. A cylindrical jar contains 100 kg of mercury. Estimate the height of the mercury to the nearest centimetre, given that the inside diameter of the jar is 12 cm and that 1 cc of mercury weighs 13·5 gm. (Take $\pi = 3·14$.)

(N.C.T.E.C.)

37. The circumference of a certain solid cylinder is equal to half its length. Obtain formulæ for its volume in terms of:

(1) its diameter, represented by d in.;
(2) its length, represented by l in.

If the volume of the cylinder is $\dfrac{27}{2\pi}$ cu in., what is its length? (N.C.T.E.C.)

38. A cast-iron weight should be 5 lb, but weighs 4·98 lb. To correct this a hole $\frac{3}{8}$ in. in diameter is drilled in the weight and then plugged with lead. Calculate in inches to three significant figures how deep the hole should be.

The weights of 1 cu in. of cast iron and of lead are 0·26 lb and 0·41 lb respectively. (U.E.I.)

39. A tube 50 cm long of small bore was filled with mercury which was afterwards run out and weighed. Weight of mercury = 33·99 gm.

Calculate:

(1) the mean cross-sectional area of the bore of the tube;

(2) the diameter of the bore.

(1 cc of mercury weighs 13·6 gm.)

(U.E.I.)

40. An iron bar 5 ft long has a uniform cross-section in the form of a sector of a circle. The angle subtended by the arc is 65° and the radius of the arc is $3\frac{1}{2}$ in.

Given that the iron weighs 480 lb per cu ft, find the weight of the bar. (U.E.I.)

41. The length of a hexagonal bar of mild steel is 16 ft. The perimeter of its cross-section is 7·5 in. Find the weight of the bar given that the volume of 1 lb of the steel is 35·7 cu in. (N.C.T.E.C.)

42. If a cubic foot of iron weighs 480 lb, find the weight per square foot of iron plating $\frac{1}{4}$ in. thick. (U.L.C.I.)

CHAPTER 13

VARIATION

(A) When one Quantity is Directly Proportional to Another

1. If I go into a shop to buy tea of a certain quality, I know that the amount I must pay is proportional to the weight I buy, whatever the price per lb.

Thus, if I buy 4 lb, I know I must pay twice as much as for 2 lb. The ratio of costs for two different amounts will be the same as the ratio of their weights. If we generalise this and represent two weights by W_1, W_2 and the corresponding costs by C_1, C_2, then we know that

$$\frac{C_1}{C_2} = \frac{W_1}{W_2}$$

Thus any two such pairs of values gives us four numbers in proportion, and so we say that the cost is proportional to the weight, or, more precisely, the **cost is directly proportional to the weight.**

Similarly if a train is moving with uniform velocity the distance passed over is dependent on the time. Thus we know that the distance passed over in 7 sec would be $3\frac{1}{2}$ times the distance passed over in 2 sec. If two times are represented by T_1, T_2 and the corresponding distances by S_1, S_2,

then $$\frac{S_1}{S_2} = \frac{T_1}{T_2}$$

As before, the **distance is directly proportional to the time.**

2. Using another method of expressing the same idea, which is common in Mathematics, we say,

Distance varies directly as time.

In the first example we can say,

Cost varies directly as weight.

To take another example, we know that the circumference of a circle is proportional to the diameter.

Thus if we have two circles of circumferences C_1 and C_2 and diameters d_1 and d_2

then $$\frac{C_1}{C_2} = \frac{d_1}{d_2}$$

The **circumference varies directly as the diameter.**

The last example might also be written:

$$\frac{C_1}{d_1} = \frac{C_2}{d_2}$$

If we had another circle of circumference C_3 and diameter d_3 we could similarly write

$$\frac{C_1}{d_1} = \frac{C_2}{d_2} = \frac{C_3}{d_3}$$

These and similar ratios for other circles must all have the same constant value.

Let this value be represented by K, and let C and d represent the circumference and diameter of any circle, then it follows that

$$\frac{C}{d} = K$$

whence $$C = K \, . \, d$$

Similarly, using general notations for the examples above we could write

$$C = K \, . \, W$$

and $$S = K \, . \, T$$

It should be carefully noted that the K has a different value in each case. The student knows that in the case of the circle $K = \pi$ and the equation becomes

$$C = \pi d$$

The symbol $\propto$ is used in mathematics to represent "**proportional to**" or "**varies as.**"

Thus we could express the relations between the quantities above as

$$\mathbf{C \propto W}$$
$$\mathbf{S \propto T}$$
$$\mathbf{C \propto d}$$

To generalise the above:

If a quantity y is proportional to, or varies directly as another quantity x,

then $$y \propto x$$
and $$y = K \, . \, x,$$

where K is a constant, the value of which *depends on* the quantities considered.

3. Geometrical Illustration

If we plot circumference against diameter, we get a straight-line graph.

If, as in Fig. 118, we represent the lengths of circumferences C_1, C_2, C_3, corresponding to diameters d_1, d_2, d_3,

then $$\frac{C_1}{d_1} = \frac{C_2}{d_2} = \frac{C_3}{d_3}.$$

We have seen also (Chapter 10, § 8) that each of these ratios represents tan θ, where θ is the angle made by the straight line with the x axis.

Clearly the graphical re-

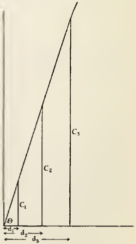


FIG. 118.

presentation of such a relation as one quantity varying directly as another, is a straight line.

We have also seen that the equation of a straight line passing through the origin is $y = mx$, so that m, the tangent of the angle made with the x axis, represents the constant K, which we have used above.

4. Other Forms of Variation

(1) We have seen that the circumference of a circle varies directly as the diameter. But this is not true of the area of the circle. If two circles have areas A_1 and A_2, diameters d_1 and d_2,

then $$A_1 = \pi \cdot \frac{d_1^2}{4}$$

and $$A_2 = \pi \cdot \frac{d_2^2}{4},$$

whence $$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2}$$

Similarly for other circles. Generally we may say that the **Area is proportional to the square of the diameter,** or

$$A \propto d^2$$

Hence $$A = K \cdot d^2,$$

where $$K = \frac{\pi}{4}.$$

Similarly with a falling body—the velocity not being uniform—we learn in Mechanics that the **distance passed over is proportional to the square of the time,** or if S and T be the distance and time

$$S \propto t^2.$$

The actual formula which connects them is

$$S = \tfrac{1}{2}gt^2.$$

where $\frac{1}{2}g$ represents the constant K.

(2) Let us consider two spheres whose volumes are represented by V_1 and V_2 and whose radii are r_1 and r_2.

Then $$V_1 = \tfrac{4}{3}\pi r_1^3$$

and $$V_2 = \tfrac{4}{3}\pi r_2^3$$

whence $$\frac{V_1}{V_2} = \frac{r_1^3}{r_2^3}$$

We can deal similarly with other spheres, and generally we can say that the **volume is proportional to the cube of the radius,** or

$$V \propto r^3$$

Hence as before $$V = Kr^3$$

and $$K = \tfrac{4}{3}\pi$$

(3) The time of vibration of a simple pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum and g is a constant at the point on the earth's surface where the experiment is carried out.

This formula can be written in the form

$$T = 2\pi \cdot \frac{l^{\frac{1}{2}}}{g^{\frac{1}{2}}}$$

If therefore we have two pendulums whose lengths are l_1 and l_2 and whose times of vibration are T_1 and T_2

$$\frac{T_1}{T_2} = \frac{l_1^{\frac{1}{2}}}{l_2^{\frac{1}{2}}}$$

We can deal similarly with pendulums of other lengths. Hence we may say that the **time of vibration is proportional to the square root of the length,** or

$$T \propto l^{\frac{1}{2}}$$

or $$T \propto \sqrt{l}$$

Therefore as in the previous cases $T = K \cdot \sqrt{l}$.

In this case the constant $K = \dfrac{2\pi}{g^{\frac{1}{2}}}$

(B) Inverse Variation

5. The cases so far dealt with have been examples of **direct variation.**

If, however, we compare two rectangles which have the **same area,** we know that as one dimension increases in magnitude the other dimension will decrease. If one has double the height of the other, its base will be one-half that of the other.

If the length of one is $2\frac{1}{2}$ times the length of the other, its base will be only $\frac{2}{5}$ of that of the other rectangle, and so on.

Let h_1 and h_2 be the heights of rectangles of equal area, and let b_1 and b_2 be their bases.

Then $$h_1b_1 = h_2b_2$$

or $$\frac{h_1}{h_2} = \frac{b_2}{b_1}$$

or $$\frac{h_1}{h_2} = \frac{\dfrac{1}{b_1}}{\dfrac{1}{b_2}}$$

This relation is expressed by stating that the heights are **inversely** proportional to the bases, or that the height of a rectangle **varies inversely** as its base, *provided its area remains the same.*

Generalised, we say that

$$h \propto \frac{1}{b}$$

$$\therefore \quad h = K \cdot \frac{1}{b}$$

where K is a constant.

Many examples could be given of this particular type, but two will suffice.

(*a*) The **volume** of a gas varies **inversely** as the **pressure** if the temperature be constant.

If P = the pressure,
V = the volume,

then $$V \propto \frac{1}{P}$$

so that $$V = K \cdot \frac{1}{P}$$

where K is a constant depending on the mass of gas employed.

(*b*) The electrical resistance of a wire of given length and material is **inversely proportional** to the area of its cross-section.

If A_1 and A_2 are the cross-sections of two such wires and R_1 and R_2 are the corresponding resistances, then

$$\frac{R_1}{R_2} = \frac{\dfrac{1}{A_1}}{\dfrac{1}{A_2}}$$

$$= \frac{A_2}{A_1}$$

and similarly for other cross-sections.

Generalising, we say that

$$R \propto \frac{1}{A}$$

Hence $$R = K \cdot \frac{1}{A}$$

where K is a constant depending on the material of which the wire is made.

(C) Determination of the Quantity K

6. The problem which usually confronts the student is to determine the constant K, and so obtain the law connecting the quantities concerned.

To enable us to do this, we must in general know two corresponding values of the quantities involved in the variation.

Having determined K, we can employ it, if necessary, to find the value of one of the variables when the other is known.

The following examples will illustrate the method employed.

Example 1. *The area of a triangle varies as its height if the base is unaltered. If the area of a triangle be* 18·6 *sq in. when its height is* 4·5 *in., what is the area of a triangle on the same base when the height is* 2·4 *in.?*

Let $A \equiv$ the area
$h \equiv$ the height.

Then $A \propto h$

that is $A = K \cdot h$

where K is a constant.

Substituting the values given

$$18{\cdot}6 = K \cdot 4{\cdot}5$$

$$\therefore \quad K = \frac{18{\cdot}6}{4{\cdot}5}$$

Hence the relation between A and h is expressed by

$$A = \frac{18{\cdot}6}{4{\cdot}5} \cdot h$$

$$\therefore \quad \text{When } h = 2{\cdot}4, \; A = \frac{18{\cdot}6}{4{\cdot}5} \times 2{\cdot}4$$

$$= 9{\cdot}92 \text{ sq in.}$$

If the student will carefully examine the working of this example he will see that the constant K is in fact $\frac{18{\cdot}6}{4{\cdot}5}$ **inches** in order to lead to the answer A = 9·92 **square inches.**

Example 2. *The resistance of a given length of wire* **varies inversely** *as the area of its cross-section.*

If the resistance of a piece of wire of 0·015 sq mm cross-section be 3·6 ohms, what is the resistance of a piece of wire of the same length whose cross-section is 0·0063 sq mm?

Let $A \equiv$ the cross-section
$R \equiv$ the resistance.

Then $R \propto \frac{1}{A}$

that is, $R = K \cdot \frac{1}{A}$

Substituting the values given,

$$3{\cdot}6 = K \frac{1}{0{\cdot}015}$$

$$\therefore \quad K = 3{\cdot}6 \times 0{\cdot}015.$$

Hence the relation between R and A is expressed by

$$R = \frac{3{\cdot}6 \times 0{\cdot}015}{A}$$

$$\text{If } A = 0{\cdot}0063 \text{ sq mm, } R = \frac{3{\cdot}6 \times 0{\cdot}015}{0{\cdot}0063}$$

$$= 8{\cdot}57 \text{ ohms.}$$

In this example the "dimensions" of K are clearly Area × Resistance.

7. In some cases, however, a variable quantity may depend on two or more other variables

For example, if $A \equiv$ the area of a triangle, $h \equiv$ its height and $b \equiv$ its base, we know that:

(1) $A \propto h$ if the base is the same.
(2) $A \propto b$ if the height is the same.

We also know that $A = \frac{1}{2}bh$
or $A = K \,.\, bh$
where $K = \frac{1}{2}$.

In other words, $A \propto bh$

Hence we can say that $A \propto bh$ when **both** b and h vary.
Generally if x varies as y when p is constant and x varies as p when y is constant, then x varies as the product of p and y when both p and y vary. I.e. $x \propto py$. Hence $x = K \,.\, py$.

Example 1. *The force between two magnetic poles varies jointly as their strengths and inversely as the square of the distance between them. If two poles of strengths of* 8 *and* 6 *units repel one another with a force of* 3 *dynes when placed* 4 *cm apart, with what force will two poles whose pole strengths are* 5 *and* 9 *units repel one another when* 2 *cm apart?*

Let $F \equiv$ the force, m_1 and m_2 the pole strengths and d the distance apart. Then F varies jointly as the product of m_1 and m_2 and inversely as d^2.

or $$F \propto \frac{m_1m_2}{d^2}$$

that is, $$F = K \,.\, \frac{m_1m_2}{d^2}$$

$$\therefore \quad K = \frac{3 \times 16}{8 \times 6} = 1$$

$$\therefore \quad F = \frac{m_1m_2}{d^2}$$

Hence in the second case $F = \frac{5 \times 9}{4} = 11{\cdot}25$ dynes.

Example 2. *The number of heat units* (H) *generated by an electric current varies directly as the time t and the square of the voltage* E, *and inversely as the Resistance* R.

If $H = 60$ *when* $t = 1$, $E = 100$, *and* $R = 40$, *find*

(1) *The value of* H *when* $E = 200$, $R = 120$ *and* $t = 300$.
(2) *The value of* t *when* $E = 120$, $R = 90$ *and* $H = 5760$. (U.L.C.I.)

(1) From the question

$$H \propto \frac{t \,.\, E^2}{R}$$

then $$H = K \,.\, \frac{tE^2}{R}$$

where K is a constant.

There are four variables involved here.
Hence, substituting the values given, we have

$$60 = K \,.\, \frac{1 \times 100^2}{40}$$

$$\therefore \quad K = \frac{60 \times 40}{100^2} = \frac{24}{100} = 0{\cdot}24$$

The actual relation, then, between the four quantities is expressed by

$$H = \frac{0{\cdot}24tE^2}{R}$$

$$\therefore \quad \text{The required value of } H = \frac{0{\cdot}24 \times 300 \times 200^2}{120}$$
$$= 24{,}000 \text{ units.}$$

(2) Since
$$H = \frac{0{\cdot}24tE^2}{R}$$
$$H \,.\, R = 0{\cdot}24tE^2$$
$$\therefore \quad t = \frac{HR}{0{\cdot}24E^2}$$

Substituting for H, R and E we have:

$$t = \frac{5760 \times 90}{0{\cdot}24 \times 120 \times 120}$$
$$\therefore \quad t = 150 \text{ sec.}$$

EXERCISE XIII

1. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. A copper wire 0·08 in. in diameter and 1000 yd long has a resistance of 4·84 ohms. What is the resistance of a copper wire 0·04 in. in diameter and 100 yd long? (Sunderland.)

2. The intensity of illumination given by a lamp varies directly as the candle-power of the lamp and inversely as the square of the distance of the lamp from the screen. If a lamp 40 ft from a screen produces the same intensity of illumination as a lamp of 10 candle-power placed 10 ft from the screen, find the candle-power of the first lamp. (Coventry.)

3. (*a*) The horse-power of a windmill varies directly as the total sail area and the cube of the velocity of the wind. If the sail area is 1000 sq ft and the wind velocity 15 m.p.h. the horse-power is 9·7. Find the horse-power if the sail area is 1200 sq ft and the wind velocity 20 m.p.h.

(*b*) Two candlesticks are the same shape. It costs 15*s*. to gild the smaller, which is 6 in. high. What is the cost of gilding the other, which is 15 in. high? (Nuneaton.)

4. (i) Assuming the speed of flow to be constant, what diameter of pipe will pass six times as great a volume of water as a pipe $1\frac{1}{4}$ in. diameter?

(ii) The deflection of a bar being turned between lathe centres varies directly as the cube of the length of the bar and inversely as the fourth power of the diameter of the bar.

A 4 in. dia bar 1 ft long is found to be deflected 0·0025 in. How much will a bar 2 in. dia and 2 ft long be deflected by an equivalent cut? (Coventry.)

5. The deflection, y, of a beam is proportional to the load (W) and to the cube of the length (L); and inversely proportional to the cube of the depth (d) of the beam. Write a formula for y in terms of W, L and d.

For a certain beam the deflection is 2 in. What would be the deflection if the load was doubled, the length reduced to a quarter of its original value and the depth halved? (Burton upon Trent.)

6. (*a*) The areas of similar plane figures are proportional to the squares of corresponding lengths. Write down a corresponding statement regarding the volumes of similar solids.

(*b*) A cylindrical measure has a height of $5\frac{1}{2}$ in. and holds a pint. What must be the height of an exactly similar measure holding a gallon?

(*c*) A marquee 250 ft long requires 8000 sq yd of canvas. How much would be required for an exactly similar one 300 ft long? (E.M.E.U.)

7. A marine engine has three cylinders whose diameters are in the ratio 3 : 5 : 8. The diameter of the smallest cylinder is 12 in. Find the other two diameters and the ratio of the cylinder volumes. The three cylinders have the same length. (N.C.T.E.C.)

8. The diameter (d) of a shaft is proportional to the cube root of the horse-power (H) it is required to transmit. If the diameter necessary to transmit 12 h.p. is 2 in., find the formula which connects them.

What horse-power can be transmitted by a shaft of 3 in. diameter?

9. The time of vibration of a simple pendulum is proportional to the square root of its length.

Assuming that one which beats seconds is 39 in. long, what will be the time of one vibration if its length is increased by 3 in.?

10. For a given source of light the intensity of illumination (I) is inversely proportional to the square of the distance (d). A surface is illuminated with a certain intensity when at a distance of 5 ft. At what distance must the surface be placed so that the intensity of illumination is $1\frac{1}{2}$ times as great?

11. The extension of a rubber cord is directly proportional to its length (L) and to the load applied (W), if the cross-section and material be the same.

If a cord of length 3 ft is stretched 3 in. by a load of $2\frac{1}{2}$ lb, what extension will be produced in a cord 2 ft long by a load of $3\frac{1}{2}$ lb?

12. When a gas expands at constant temperature its pressure varies inversely as its volume. When the pressure is 90 lb per sq in., the volume is 1·8 cu ft. Find the pressure to the nearest pound per square inch when the volume is 2·5 cu ft; and the volume to the nearest hundredth of a cubic foot when the pressure is 75 lb per sq in.

13. The electrical resistance (R) of a wire varies as $\frac{L}{d^2}$, where L is the length and d is the diameter. The weight (W) of the wire varies as Ld^2. Show that the resistance of a wire varies as W/d^4. If a pound of wire of diameter 0·06 in. has a resistance of 0·25 ohm, what is the resistance of a pound of wire of the same material the diameter being 0·01 in.?

14. Assuming that the velocity of a falling body is proportional to the square root of height fallen through, and that after falling through a height of 1 ft the speed is 8·025 ft per sec, find to within $\frac{1}{10}$ ft per sec what the speed will be after falling through 873·4 ft.

15. The resistance of a wire varies directly as its length and inversely as its sectional area. If the resistance of 500 yd of copper wire of diameter 0·028 in. is 19 ohm, find the resistance of 1 mile of similar wire 0·16 in. in diameter. (U.L.C.I.)

16. The load that a beam of given depth will carry is directly proportional to the breadth and inversely proportional to the length, the depth being constant.

If a beam of length 7 ft and width $1\frac{3}{4}$ in. can support a load of 4 tons, what load can be supported by a beam 5 ft long and $2\frac{1}{2}$ in. wide, the depth and the material being the same?

17. The load raised by a winding engine varies directly as the steam pressure and inversely as the diameter of the winding drum. If a load of 45 cwt is raised by a drum of 10 ft diameter when the steam pressure was 90 lb per sq in., what load should be raised by a drum of 12 ft diameter if the steam pressure is 75 lb per sq in.? (U.L.C.I.)

18. The price of a certain range of cable sizes is directly proportional to the length and to the cross-section of the copper. Find the cost of a 100-metre coil of cable of cross-section 1·25 sq mm, if the cost of 110 yd of cable of diameter 0·044 in. is 15s. Take 1 m = 39·37 in. and 1 sq in. = 6·45 sq cm. (U.L.C.I.)

19. The horse-power of the engines of a ship being proportional to the cube of the speed, find the speed when the horse-power is 8000 if the horse-power is 2000 at a speed of 10 knots.

CHAPTER 14

MORE DIFFICULT GRAPHICAL WORK

1. Chapter 7 was devoted to a consideration of graphs generally, and in particular to the study of the **straight-line graph** and its Law. In that chapter, Fig. 28 provided us with an example of a curve, which also seemed to follow some law, and it is now our purpose to study some well-defined curves, which are based on definite laws, and show the relation between the **independent** and **dependent** variables.

2. The Curve of Squares

The simplest of the above-mentioned curves is the **curve of squares,** and the law connecting the two variables is usually given in the form

$$y = x^2$$

To draw the curve it will be sufficient to take values of x from 0 to $+4$, and from 0 to -4. Find the corresponding values of y and set out as shown in the table below.

If $x =$	0	$\frac{1}{2}$	1	1·4	2	2·5	3	4	−1	$-\frac{1}{2}$	−1·4	−2	−2·5	−3	−4
$y =$	0	$\frac{1}{4}$	1	1·96	4	6·25	9	16	1	$\frac{1}{4}$	1·96	4	6·25	9	16

The points are plotted and the curve drawn as shown in the accompanying figure (Fig. 119).

An examination of this table shows that when values of x equal in magnitude but opposite in sign are taken, the corresponding values of y are equal.

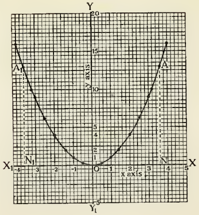


FIG. 119.

Thus if $x = +1{\cdot}4$ or $-1{\cdot}4$, $y = +1{\cdot}96$.

Consequently for every point on the curve to the right of the y axis there is a corresponding point to the left of the axis. If the curve be folded about that axis, the two parts will coincide. The curve is therefore **symmetrical about the y axis.**

Similarly corresponding to any value of y there are two values of x, equal in magnitude, but opposite in sign.

Thus when $y = 9$, the corresponding values of x are $+3$ and -3.

Draw a line across the graph such as A_1A parallel to the

x axis, and draw perpendiculars A_1N_1 and AN to the x axis.

At A and A_1, $y = 13$ — that is, $x^2 = 13$.

Corresponding to this value of y, x is represented by $ON = +3\cdot6$ and by $ON_1 = -3\cdot6$.

Actually this step provides us with a method of determining the square roots of numbers within the compass of the graph, and which themselves have not been specially plotted.

NOTE.—This curve is called a **parabola.**

3. Choice of Scales in Graphical Work

1. Choice of Units. It will be noticed that in drawing the curve of $y = x^2$ (Fig. 119) different units were taken on the two axes. The object of this was to obtain a graph which is more satisfactory for practical purposes. In this graph the values of y increase more rapidly than those of x. Consequently if the same units were employed on both axes very little of the curve could be shown, within the limits of the paper. If the curve is to be of practical value (a) it should be drawn as large as the paper will allow, (b) the units taken should be as large as possible. When, therefore, the tables of values of x and y has been made, the units to be used on each axis should be selected so that the curve may be drawn to the best advantage.

2. Position of Axes. Similarly before drawing the axes for the curves, an examination should be made of the tables of values. If, for example, there will be no negative values of y, as in $y = x^2$, the x-axis should be drawn near the bottom of the paper. Similarly there may be no negative values of x in some cases. Then the y axis should be drawn well to the left of the paper.

4. The curves of $y = ax^2$ and $y = ax^2 + b$

The curves of such equations as $y = x^2 + 3$, $y = x^2 - 2$ will be of the same shape as $y = x^2$, but the lowest point will not be at the origin. For example, if we consider $y = x^2 + 3$, since every value of x^2 is increased by 3 to obtain the value of y, then the lowest point of the curve will be at the point $+3$ on the y-axis.

Similarly the lowest point of $y = x^2 - 2$ will be at the point -2 on the y axis. Generally the curve of $y = x^2 + b$ will be a parabola with the lowest point at b on the y axis.

5. The curve of $y = ax^2$, where a is any positive number, will also be a parabola, symmetrical about the y axis and with its lowest point at the origin.

Generally, using the same argument as above, the curve of $y = ax^2 + b$ will be a parabola symmetrical about the y axis and with its lowest point at b on this axis.

6. If the equation includes a term of the first degree in x, *i.e.* is of the form

$$y = ax^2 + bx + c$$

it will be found that the graph is still a parabola, but the axis of symmetry will not be the y axis, but a line parallel to it.

To draw a graph whose law is of the form $y = ax^2 + bx + c$, where a, b and c are constants.

Let $y = 2x^2 - 3x - 5$ in which $a = 2$, $b = -3$, and $c = -5$.

In assuming a value of x and calculating the corresponding value of y, the student is recommended to adopt some such plan as that set out on following page.

If $x =$	0	$\frac{1}{2}$	1	2	3	4	5	-1	-2	-3	-4
$2x^2 =$	0	$\frac{1}{2}$	2	8	18	32	50	2	8	18	32
$-3x =$	0	$-1\frac{1}{2}$	-3	-6	-9	-12	-15	3	6	9	12
-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
$y =$	-5	-6	-6	-3	4	15	30	0	9	22	39

The points showing the relation between x and y are then plotted as shown in Fig. 120.

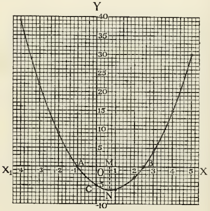


Fig. 120.

If the student finds, on trying to draw the curve, that he is in doubt about its true form at any part, he should work out further values of y for intermediate points.

The second column gives an example of this where $x = \frac{1}{2}$.

It will be observed, as in the previous cases, that the constant $c = -5$ is represented by the intercept on the y axis, since it is the value of y when $x = 0$.

7. Significance of the Intersection of the Curve with the x Axis

It must be remembered that the vertical distance of any point on the curve, measured above or below the x axis according to the vertical scale, represents a value of y positive or negative, corresponding to a definite value of x.

In Fig. 120, the curve cuts the x axis at A and B so that at these points $y = 0$.

At A, $x = -1$, so that when $y = 0$, $x = -1$.

At B, $x = 2{\cdot}5$, so that when $y = 0$, $x = 2{\cdot}5$.

But y represents the expression $2x^2 - 3x - 5$, so that we can say:

When $2x^2 - 3x - 5 = 0$, $x = -1$ or $2{\cdot}5$.

In other words, these values of x satisfy the equation $2x^2 - 3x - 5 = 0$.

They are therefore its roots.

It follows, then, that if we desire to solve graphically an equation of the form $ax^2 + bx + c = 0$, we may draw a graph to represent the varying values of the **expression** $\boldsymbol{ax^2 + bx + c}$, and note the points of intersection of this graph with the x axis.

The values of x at these points will give the desired roots.

This, as we shall see later, is only one of the methods we can employ.

8. Turning Point and Minimum Value

There is one other interesting point to note with regard to this graph.

In Fig. 120 the point N marks the lowest point and also the **turning** point of the graph.

Taking values from the graph we find that $MN = 6\frac{1}{8}$ gives the **minimum value** of the expression $2x^2 - 3x - 5$, and the corresponding value of x is 0·75.

It should also be noted that the axis of symmetry passes through this point.

9. Alternative Graphical Method of solving an Equation of the form $ax^2 + bx + c = 0$

We will take the equation $2x^2 - 3x - 5 = 0$ which has been dealt with above. This is the same as solving the equation $2x^2 = 3x + 5$. In other words, to solve this

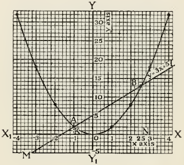


Fig. 121.

equation we have to find the values of x when the expression $2x^2$ is equal to the expression $3x + 5$.

Hence let $y = 2x^2$, and let $y = 3x + 5$.

Then draw the graph for each, and note the points of intersection.

The two graphs are shown in Fig. 121 and they intersect at the points A and B. Draw BN and AK perpendicular to the x axis. Since B is on the straight line, BN represents a value of $3x + 5$. It also represents the value of $2x^2$, for the same value of x, viz. 2·5. Hence the value of $3x + 5$ is equal to the value of $2x^2$ when $x = 2·5$.

In other words, $2x^2 = 3x + 5$ when $x = 2·5$.

$\therefore \quad x = 2·5$ is a root of the equation

$$2x^2 = 3x + 5.$$

Similarly since the co-ordinates of A, the other point of intersection, are $(-1, 2)$, $x = -1$ satisfies both of the expressions $2x^2$ and $3x + 5$, and this value of x is therefore another root of the equation

$$2x^2 = 3x + 5$$

or

$$2x^2 - 3x - 5 = 0$$

These results are seen to be identical with those obtained by the previous method.

10. The Graph of $y = -x^2$

We have already seen that whatever value we give to x, x^2 is a **positive** quantity.

Hence $-x^2$ will always be a **negative** quantity, and therefore y is always negative.

This means that the whole of the graph must lie below the x axis.

Fig. 122 shows the graphs of $y = x^2$ and $y = -x^2$, so that they can easily be compared.

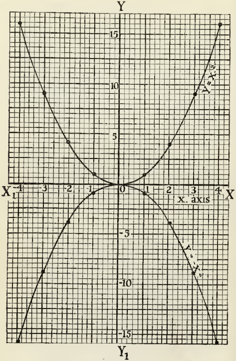


FIG. 122.

NOTE.—(1) The **highest** point of the curve $y = -x^2$ is the **turning point** at the origin and at that point is the maximum value.
(2) It is symmetrical with regard to the y axis.
(3) If we imagine the curve of $y = x^2$ to move out from the plane of the paper and rotate about the x axis through an angle of 180°, *i.e.* until it is in the plane of the paper again, it will be co-incident with the curve of $y = -x^2$.

Similarly the graph such as $y = -2x^2 - 3$ and of

$$y = -4x^2 + 10$$

could be obtained by rotating the curves of $y = 2x^2 + 3$ and $y = 4x^2 - 10$ about the x axis through 180°.

11. The Graph of $y = ax^2 + bx + c$ when a is Negative

Consider $\quad y = -x^2 - x + 12$

or $\quad y = 12 - x - x^2$

Proceeding as already shown for Fig. 120 we obtain values of x as set out below.

If $x =$	0	1	2	3	4	5	−1	−2	−3	−4	−5	−6
$y =$	12	10	6	0	−8	−18	12	10	6	0	8	−18

Fig. 123 shows the corresponding graph. The points of intersection of this graph with the x axis are at A and B, where $x = -4$ and $x = +3$, respectively.

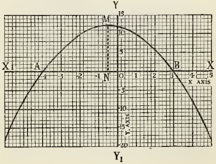


FIG. 123.

As explained for Fig. 120, these values of x are the solutions of the corresponding equation:

$$12 - x - x^2 = 0$$

or $$-x^2 - x + 12 = 0$$

12. Turning Point and Maximum Value

In this case the **turning point** M gives $MN = +12\frac{1}{4}$ as the **maximum** value of the expression $12 - x - x^2$, and the corresponding value of x is $-\frac{1}{2}$.

As in the previous cases, this graph could be obtained by rotating the curve of $x^2 + x - 12$ about the x axis through an angle of 180°.

We see, then, that when the coefficient of x^2 in a quadratic expression is **positive** there is a **minimum value** for that expression, but when the coefficient is **negative** there is a **maximum value** for the expression.

Example. *The distance of a body from the ground when projected vertically upwards with a certain velocity is given by*

$$s = -16t^2 + 128t$$

where $t \equiv$ *the time, and* $s \equiv$ *the distance. Find graphically after what time its distance will be* 156 *ft.*

We have to discover when $156 = -16t^2 + 128t$, or, in other words, when $-16t^2 + 128t - 156 = 0$.

Hence draw the graph for this expression, taking t on the horizontal axis.

Let $$y = -16t^2 + 128t - 156.$$

Table of Values.

If $t =$	0	1	2	3	4	5	6	7
$-16t^2 =$	0	-16	-64	-144	-256	-400	-576	-784
$128t =$	0	128	256	384	512	640	768	896
-156	-156	-156	-156	-156	-156	-156	-156	-156
$y =$ Expression $=$	-156	-44	36	84	100	84	36	-44

Fig. 124 shows the graph.

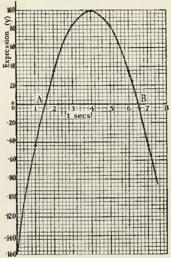


FIG. 124.

At A and B the value of the expression is zero.

At A, $t = 1{\cdot}5$ sec and at B, $t = 6{\cdot}5$ sec.

Hence the equation $-16t^2 + 128t - 156 = 0$ is satisfied when $t = 1{\cdot}5$ and $t = 6{\cdot}5$,

or $$156 = -16t^2 + 128t \text{ when } t = 1{\cdot}5 \text{ or } 6{\cdot}5 \text{ sec.}$$

Thus, after 1·5 sec, and again after 6·5 sec, the body will be 156 ft away.

13. The Graph showing the Relation between x and y when the Law is of the Form $y = \frac{a}{x}$, a being a Constant

The simplest example of this occurs when $a = 1$ so that $y = \frac{1}{x}$, and this we now proceed to draw.

Table of Values of x and y.

If $x =$	1	2	4	0·1	0·2	0·25	−1	−2	−4	−0·1	−0·2	−0·25
$y =$	1	0·5	0·25	10	5	4	−1	−0·5	−0·25	−10	−5	−4

As this table shows, when x is very small the values of y are correspondingly large, and vice versa.

On the x axis take 1 in. to represent 1 unit, and on the y axis take 0·2 in. to represent 1 unit, and plot the points indicated by the values above.

It will be observed that the curve (Fig. 125) has one branch corresponding to the positive values of x and one for the negative.

Also as x becomes less and less and approaches zero the value of y becomes greater and greater and the curve approaches nearer and nearer to the y axis, so that we have the conception of the curve meeting the y axis at an infinite distance when $x = 0$.

Similarly as x becomes greater and greater the curve approaches nearer and nearer to the x axis, so that we have the conception of it meeting the x axis when at an infinite distance.

Curves of this type are obtained when we show the relation between

(1) Volume and Pressure of a gas at constant temperature.

(2) Current and Resistance of a circuit with constant E.M.F.

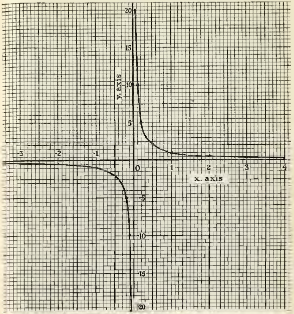


Fig. 125.

14. Average Height of a Curve from a Fixed Axis, and the Area Enclosed between the Curve and the Axis

It would be well to refer again to the discussion of the indicator diagram in § 17 of Chapter 3, and the exercise based upon it.

At the end of Chapter 2 reference was made to an irregular area such as is enclosed between a curve and a straight line; and the Mid-Ordinate Rule, which is one of the methods which is frequently utilised to find sucb an area, was explained there.

As the student is now more conversant with the methods of drawing graphs, a concrete example in which the above rule is employed, is set out below.

Example. *The following table gives related values of* P *and* V.

V . .	2	3	4	5	6	7
P . .	50	33·3	25	20	16·7	14·3

Plot the curve connecting P *and* V *and determine the area between the curve, the axis of* V *and the end ordinates. If the mid-ordinate rule is used, the mid-ordinate should be clearly shown.* (U.L.C.I.)

An examination of the above quantities shows a much wider range of values for P than for V.

Hence it is found convenient to take 1 in. to represent one unit of V on the horizontal axis, whereas on the vertical scale 1 in. represents 10 units of P.

Also, in order to place tbe graph as centrally as possible, the line $V = 2$, which is one of the end ordinates, is used to denote the scale for P. The points are then plotted according to the above data, and the smooth curve shown in the diagram is drawn through them (Fig. 126).

The diagram is then divided into ten strips of equal width, and their mid-ordinates are indicated by the dotted lines.

The sum of these mid-ordinates is

$(44 + 36 + 30{\cdot}8 + 26{\cdot}5 + 23{\cdot}5 + 21 + 19 + 17{\cdot}3 + 15{\cdot}8$
$+ 14{\cdot}8)$ units of $P = 248{\cdot}7$ units of P.

Their average is $\dfrac{248{\cdot}7}{10} = 24{\cdot}87$ units of P.

This average mid-ordinate can be taken as the **average height of the curve,** and as such, also represents the height (measured according to the vertical scale) of a rectangle whose base is AB, and whose area is equal to the area enclosed between the curve and the V axis.

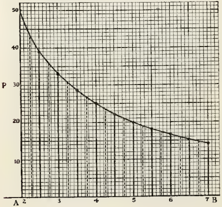


FIG. 126.

Hence, since $AB = 5$ units, the area required is

$(24{\cdot}87 \times 5) = 124{\cdot}35$ sq units.

15. Graphs of Corresponding Areas and Volumes

We have seen: (i) that corresponding surface areas occurring in similar figures vary as the squares of corresponding lengths; and (ii) that corresponding volumes

occurring in similar figures vary as the cubes of corresponding lengths.

It follows that if for any group of similar figures we plot corresponding areas against corresponding lengths the curve obtained will be a curve of squares, that is a parabola. If we plot corresponding volumes the curve will be a curve of cubes. The particulars of the figures studied will decide the scales.

Example. We select as a range of similar solids engineers' hexagon nuts. If they are similar all sizes can be represented by the same drawing. The size will in the ordinary way be indicated by the diameter of the bolt upon which the nut is to be used: this is a length.

Let us take as our standard (or datum) a 1-in. galvanised nut; and suppose that it has been "costed" as follows:

Material	1*d.*
Machining	2*d.*
Galvanising	$\frac{3}{4}$*d.*

Example. *Plot four graphs showing respectively the cost under each of these three headings, and the total cost, for nuts over a range of sizes $\frac{1}{2}$–2 in.*

Size	$\frac{1}{2}$ in.	1 in.	$1\frac{1}{2}$ in.	2 in.
(i) Material cost (pence) .	$\frac{1}{8}$	1	$\frac{27}{8}$	8
(ii) Machining cost (pence) .	$\frac{1}{2}$	2	$\frac{9}{2}$	8
(iii) Galvanising cost (pence) .	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3

In the preparation of the above table the material cost has been taken as dependent upon the volume, and therefore varying as the *cube* of the linear dimension. The machining and galvanising costs have been taken as dependent upon the surface area, and therefore varying as the *square* of the linear dimension. By addition we have:

(iv) Total cost in pence .	$\frac{13}{16}$	$3\frac{3}{4}$	$9\frac{9}{16}$	19

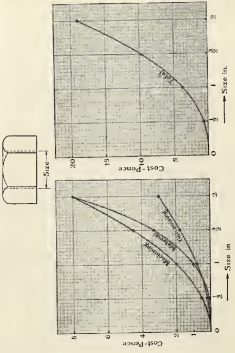


Fig. 127.

The four graphs are plotted in Fig. 127.

The machining and material curves both run from cost 0*d*. to cost 8*d*., and illustrate the difference in form between a curve of squares and a curve of cubes.

EXERCISE XIV

1. Plot a graph showing the relation between the area A of a circle in square inches and its diameter D in inches, for values of D: 0, 1, 2, 3, 4 and 5.

Calculate the areas to one decimal place only. From your graph

(1) Find the diameter when $A = 12$.
(2) Find the area when $D = 1{\cdot}5$. (U.E.I.)

2. Draw the graph of $y = 2x^2 + 7x - 4$.
Where does it cut the axis of x?

3. Solve graphically $x^2 - x - 6 = 0$. Find from your graph the values between which x must lie so that the expression $x^2 - x - 6$ may be negative.

4. If $y = x^2 + 2x - 3$, find graphically (*a*) the value of x when $y = 0$, (*b*) the minimum value of y.

5. With the values of n, given below, draw a graph showing the relation between n and n^2, taking 1 in. to represent 0·1 both for n and n^2. Obtain from your graph the values of the square roots of 0·08 and 0·15

$n = 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}5, 0{\cdot}6.$ (N.C.T.E.C.)

6. Draw the graph of $y = 4x^2 - 8x - 7$ from $x = -3$ to $x = 5$. Find the values of x which make $y = 0$. (E.M.E.U.)

7. Make a table of the values of $\sin\theta$ and $\cos\theta$ from $\theta = 0°$ to $\theta = 360°$ taking increments in θ of 30°; hence plot the graph of

$$y = \sin\theta + \cos\theta.$$

From the graph find:

(*a*) the maximum value of y;
(*b*) the values of θ between 0° and 180° for which $y = 0$ and $y = 0{\cdot}5$. (Rugby.)

8. A long strip of metal of width 12 in. is formed into an open gutter of rectangular cross-section by bending equal parts x in. of the width through 90°.

Show that the cross-sectional area A of the gutter so formed is given by $A = 12x - 2x^2$; also, by plotting the value of A for values of x from 0 to 4, find the greatest area obtainable and the value of x giving this greatest area. (Burton upon Trent.)

9. The efficiency E of a water-wheel is given by

$$E = \frac{400u(v - u)}{v^2}\ \%$$

where v is the jet velocity and u is the wheel velocity. Taking $v = 40$ ft per sec, calculate E for $w = 5, 10, 14, 18, 22$ and 25 ft per sec.

By means of a graph of E against u, estimate the maximum efficiency and the ratio of $\frac{u}{v}$ at which it occurs. (U.L.C.I.)

10. The height of a projectile above its point of projection at any time t sec is given by $h = 96t - 16t^2$ ft. Plot a graph showing the variation of height from $t = 0$ to $t = 6$ and use the graph to find:

(i) the time taken to reach maximum height;
(ii) the time for which the projectile is at a height of more than 100 ft above the level of the point of projection. (Nuneaton.)

11. Refer to the Example of § 15. A series of bolts (machined and galvanised) are of length under the head

equal to four diameters. A bolt 1 in. dia is costed as under:

Material	$2d.$
Machining	$1\frac{1}{2}d.$
Galvanising . . .	$1{\cdot}2d.$

Plot graphs showing the itemised and total costs for bolts from $\frac{3}{8}$ in. dia to $1\frac{1}{2}$ in. dia.

12. Find graphically

(1) the maximum value of $5x - x^2 + 6$,
(2) the values of x between which the expression is positive.

13. Draw the graph of $y = 6x - x^2 - 3$. From it find x when y is a maximum, and the roots of the equation $6x - x^2 - 3 = 0$.

14. Draw the graphs of $y = x^2$ and $2y = 3x + 9$ on the same diagram and deduce the roots of the equation $2x^2 - 3x - 9 = 0$.

15. Graph each of the functions $\frac{1}{2}x^2$ and $(3 - 0{\cdot}4x)$ for values of x from $-3\frac{1}{2}$ to $+3\frac{1}{2}$ using the same scales and reference axes for both graphs. By means of the graphs estimate the values of x for which $\frac{1}{2}x^2 = 3 - 0{\cdot}4x$. (N.C.T.E.C.)

16. The sum of the length and five times the breadth of a rectangle is 17·5 in.; its breadth is x in. Express in terms of x (1) its length, (2) its area.

(3) Plot its area against x for values of x from 0 to $3\frac{1}{2}$. By means of the graph estimate within $\frac{1}{8}$ in. the breadth of the rectangle when its area is a maximum. (N.C.T.E.C.)

17. The law connecting the volume (v cu ft) of water in a certain trough with the depth of the water (h ft) is $v = 3\ 2h^2$. Calculate the volumes corresponding to depths of 3, 6, 9, 12, 15, 18 in., and construct a graph by plotting v against h. From the graph estimate, within a fifth of an inch, the depth of water corresponding to volumes of 3 cu ft and 6 cu ft respectively. (N.C.T.E.C.)

18. The efficiency e of a certain type of water-wheel is given by the expression

$$e = \frac{4u(v - u)}{v^2}$$

where v is the velocity of the incoming jet of water and u the speed of the wheel.

Taking v as 30 ft per sec, calculate e for values of u equal to 10, 13, 16, 20 and 25 ft per sec and tabulate.

Plot to as big a scale as the paper will allow e vertically and u horizontally. Use the graph to find the ratio of $\frac{u}{v}$ which makes the efficiency a maximum. (U.E.I.)

19. If $x - \frac{8}{x} = y$, find the values of y for values of x from 2 to 3. Plot on squared paper and find what value of x makes $y = 0$. (U.E.I.)

20. Graph each of the functions $0{\cdot}2x^2$ and $\frac{0{\cdot}5}{x}$ for values of x from -4 to $+4$ using the same scales and reference axes for both graphs.

For what values of x are the values of these functions equal? (N.C.T.E.C.)

21. Plot the curve given by the following values of x and y. Find the area included by the curve and the axes of x and y. Also find the average height of the curve.

x . . .	6	5·5	5	4	3	2	1	0
y . . .	0	1·4	2·2	3·1	3·5	3·8	3·9	4

(U.L.C.I.)

22. The following values of the load (W) and extension (x) were obtained during a tensile test of a steel bar.

x in. .	0·27	0·48	0·75	1·1	1·5	2·0	2·5	2·8
W tons .	11·6	13·2	14·4	15·2	15·6	15·6	14·8	13·4

Plot the curve connecting W and x and determine the average value of W between $x = 0·5$ and $x = 2·5$.

(U.L.C.I.)

23. Plot the related values of x and y given in the table and join them in the order given. Find the area of the closed figure thus formed and give the answer in foot-pounds.

x ft .	0	0·3	0·4	0·5	0·6	0·7	0·6	0·1	0
y lb .	60	60	50	30	20	10	0	0	10

(U.L.C.I.)

CHAPTER 15

QUADRATIC EQUATIONS

1. In the last chapter we found it was possible to solve an equation of the form $ax^2 + bx + c = 0$ by means of a graph.

This method, however, of solving such an equation is somewhat cumbersome, and does not admit of the same degree of accuracy as can be obtained by purely algebraical methods. There are, however, certain equations of a more involved and difficult type which can be solved only by graphical methods.

An equation of the type $ax^2 + bx + c = 0$ involving x in the second degree, and containing no higher power, and in which the constants a, b and c can have any numerical values, is termed a **quadratic equation.**

We will now proceed to the algebraical methods of solution of such equations.

Case I

2. When $b = 0$, the equation becomes $\boldsymbol{ax^2 + c = 0}$. It will be remembered that in dealing with the curve of squares in the early part of the last chapter, we found that corresponding to any value of y there were two equal and opposite values of x.

Hence if $\quad y = x^2 = 25$

$\quad\quad\quad\quad x = \pm 5$

The sign $\pm$ indicates that both $+ 5$ and $- 5$ are the square roots of 25.

To take another example.

Solve $2x^2 - 9 = 0$
then $2x^2 = 9$
that is, $x^2 = 4{\cdot}5$

$$\therefore \quad x = \pm\sqrt{4{\cdot}5} = \pm 2{\cdot}12 \text{ (approx.)}$$

Again referring to the curve of squares, it will be observed that the whole of the curve is above the x axis, whatever may be the value of x, positive or negative, and therefore **y is always positive.**

Therefore x^2 is also positive, and it is only in these circumstances that we can obtain the equal and opposite values of x.

If we have the equation

$$2x^2 + 9 = 0$$
or $2x^2 = -9$
then $x^2 = -4{\cdot}5$

Now, when a number is squared, we are multiplying together two quantities with the same sign.

Consequently, in accordance with the rule of signs, the result must be a positive quantity.

This equation, then, does not admit of a solution which has any arithmetical meaning, and the roots are expressed by

$$x = \pm\sqrt{-4{\cdot}5}$$

3. Case II. When the terms involved in the equation form a perfect square

Example 1. *Solve* $(2x - 11)^2 = 25$

Then $2x - 11 = \pm 5$

Hence $2x = (11 + 5)$ or $(11 - 5)$

$$\therefore \quad x = 8 \text{ or } 3$$

Example 2. *Solve* $x^2 = 6x - 9$

Rearranged this becomes $x^2 - 6x + 9 = 0$

that is $(x - 3)^2 = 0$

$$\therefore \quad x - 3 = 0$$
or $x = 3$

4. The expression $x^2 - 6x + 9$ in Example 2 above was seen to be an exact square, viz. $(x - 3)^2$, and so a solution of the equation $x^2 - 6x + 9 = 0$ was simple.

An expression such as $x^2 - 6x + 8$, although not an exact square, could clearly be converted into one by the addition of unity. Consequently if we write

$$x^2 - 6x + 8 = (x^2 - 6x + 9) - 1$$
or $(x - 3)^2 - 1$

we change the expression into an exact square less unity. This suggests a method of solving such an equation as

$$x^2 - 6x + 8 = 0$$

Since we can write this as $(x - 3)^2 - 1 = 0$
or $(x - 3)^2 = 1$

We can then proceed as in Example 1.

As we shall see, this method can be generally applied.

Let us first consider the result given on p. 72, which tells us that

$$x^2 + 2ax + a^2 = (x + a)^2$$

The problem in solving a quadratic is, starting with an expression such as $x^2 + 2ax$, to find what must be added to it to make an exact square. Now a^2, the quantity added in this general case, is **the square of half the coefficient of** $\boldsymbol{x}$, *i.e.* half of $2a$. Hence we can obtain a rule which will apply in all cases.

Thus if we want to convert $x^2 + 10x$ into an exact square we add on the square of half the coefficient of x, *i.e.* $(5)^2$. This would produce $x^2 + 10x + 25$, which is

$$(x + 5)^2.$$

5. This device we can utilise in the solution of quadratic equations as follows.

Example 1. *Solve the equation*

$$x^2 + 8x + 12 = 0$$

It is better to rewrite this as

$$x^2 + 8x = -12$$

Now add to each side $(\frac{8}{2})^2$ or $(4)^2$.

$$\text{Then} \quad x^2 + 8x + (4)^2 = -12 + 16$$

$$\text{or} \quad (x + 4)^2 = 4$$

$$\therefore \quad x + 4 = \pm 2$$

$$\text{and} \quad x = -4 + 2 = -2$$

$$\text{or} \quad x = -4 - 2 = -6$$

$$\therefore \quad \boldsymbol{x = -2 \text{ or } -6.}$$

Example 2. *Solve the equation*

$$x^2 - 7x + 12 = 0$$

$$\text{or} \quad x^2 - 7x = -12$$

$$\text{Then} \quad x^2 - 7x + (\tfrac{7}{2})^2 = -12 + \tfrac{49}{4}$$

$$\therefore \quad (x - \tfrac{7}{2})^2 = \tfrac{1}{4}$$

$$\therefore \quad x - \tfrac{7}{2} = \pm \tfrac{1}{2}$$

$$\text{and} \quad x = +\tfrac{7}{2} + \tfrac{1}{2} = 4$$

$$\text{or} \quad x = +\tfrac{7}{2} - \tfrac{1}{2} = 3$$

$$\therefore \quad \boldsymbol{x = 4 \text{ or } 3.}$$

Example 3. *Solve the equation* $3x^2 - 7x = 20$.

Referring again to the result $(x + a)^2 = x^2 + 2ax + a^2$ we have seen that the coefficient of x, viz. $2a$, is twice the square root of a^2.

This is clearly not the case, however, when the coefficient of x^2 is *not unity*.

$$\text{Thus} \quad (2x + a)^2 = 4x^2 + 4ax + a^2.$$

Consequently we can apply the rule given in the previous example for the solution of a quadratic only when the coefficient of x^2 is *unity*.

If this is not the case, we can divide both sides of the equation by the coefficient of x^2, and thus obtain the previous form.

In the given example above, dividing by 3, the coefficient of x^2, we have:

$$x^2 - \tfrac{7}{3}x = \tfrac{20}{3}$$

Adding the square of the half-coefficient of x to each side we get:

$$x^2 - \tfrac{7}{3}x + (-\tfrac{7}{6})^2 = \tfrac{20}{3} + \tfrac{49}{36}$$

Coeff. of x $= -\frac{7}{3}$
$\frac{1}{2}$ Coeff. $= -\frac{7}{6}$
$(\frac{1}{2}$ Coeff.$)^2 = \frac{49}{36}$

$$\text{that is} \quad (x - \tfrac{7}{6})^2 = \tfrac{289}{36}$$

$$\text{Hence} \quad x - \tfrac{7}{6} = \pm \tfrac{17}{6}$$

$$\therefore \quad x = \tfrac{7}{6} + \tfrac{17}{6} = 4$$

$$\text{or} \quad x = \tfrac{6}{6} - \tfrac{17}{6} = -1\tfrac{2}{3}$$

$$\therefore \quad \boldsymbol{x = 4 \text{ or } 1\tfrac{2}{3}}$$

Example 4. *Solve the equation*

$$\frac{2x - 3}{x + 2} = \frac{x + 5}{x - 4}$$

The first step in this case is to clear the equation of fractions. This is effected by multiplying both sides by the common denominator $(x + 2)(x - 4)$.

The equation then becomes

$$(2x - 3)(x - 4) = (x + 5)(x + 2)$$

$$\text{that is} \quad 2x^2 - 11x + 12 = x^2 + 7x + 10$$

$$\text{or} \quad x^2 - 18x = -2$$

$$\therefore \quad x^2 - 18x + 81 = 81 - 2$$

$$\text{and} \quad (x - 9)^2 = 79$$

Hence $(x - 9)^2 = \pm\sqrt{79} = \pm 8{\cdot}9$ approx.

$\therefore \quad x = 9 + 8{\cdot}9 = 17{\cdot}9$

or $x = 9 - 8{\cdot}9 = 0{\cdot}1$

$\therefore \quad \boldsymbol{x = 17{\cdot}9 \text{ or } 0{\cdot}1}$

6. Solving Quadratics by Factors

The method of the **completion of the square,** which we have just dealt with, is the one most commonly employed, though in some few cases the **factor method** is quicker and easier, and particularly so when the factors are obvious.

In the majority of cases in practice, they are not.

Example 1. *Solve* $x^2 = x + 6$

Bring all the terms to the L.H. side.

Then $x^2 - x - 6 = 0$

or $(x - 3)(x + 2) = 0$

Since the product of the two factors is zero, either **one** or the **other** or **both** must be zero.

If $x - 3 = 0$, $x = 3$. If $x + 2 = 0$, $x = -2$

$\therefore \quad \boldsymbol{x = 3 \text{ or } -2}$

Example 2. *Solve* $6x^2 + 11x = 35$

that is $6x^2 + 11x - 35 = 0$

Then $(3x - 5)(2x + 7) = 0$

If $3x - 5 = 0$, $3x = 5$, $x = 1\frac{2}{3}$. If $2x + 7 = 0$, $2x = -7$, $x = -3\frac{1}{2}$

$\therefore \quad \boldsymbol{x = 1\frac{2}{3} \text{ or } 3\frac{1}{2}}$

7. Problems involving Quadratics

Example 1. *Using the formula* $\dfrac{mN - nN^2}{K} = l$, *calculate* N *if* $m = 82$, $n = 4$, $K = 12$, *and* $l = 6$.

Substituting the values given we have:

$$\frac{82N - 4N^2}{12} = 6$$

that is, $82N - 4N^2 = 72$

or $4N^2 - 82N = -72$

Dividing throughout by 4, we have:

$$N^2 - \tfrac{82}{4}N = -18$$

or $N^2 - \tfrac{41}{2}N = -18$

Completing the square by the usual method we obtain:

$$N^2 - \tfrac{41}{2}N + \tfrac{1681}{16} = \tfrac{1681}{16} - 18$$

that is, $(N - \tfrac{41}{4})^2 = \tfrac{1393}{16}$

Hence $N - \tfrac{41}{4} = \pm\dfrac{\sqrt{1393}}{4}$

$$N - \tfrac{41}{4} = \pm\frac{37{\cdot}3}{4}$$

$$\left.\begin{aligned} \therefore \quad N &= \tfrac{41}{4} + \frac{37{\cdot}3}{4} = 19{\cdot}6 \\ \text{or} \quad N &= \tfrac{41}{4} - \frac{37{\cdot}3}{4} = 0{\cdot}9 \end{aligned}\right\} \text{approx.}$$

Example 2. *The length of a rectangle exceeds the breadth by* 3 *ft. If the length be doubled and the breadth be increased by* 2 *ft, the area will be increased by* 50 *sq ft.*

Find the length of the first rectangle.

Let x ft $\equiv$ the length,

then $x - 3 \equiv$ the breadth,

and area $\equiv x(x - 3)$ sq ft.

In the second rectangle

length $= 2x$ ft,

breadth $= x - 1$ ft,

area $= 2x(x - 1)$ sq ft.

Hence $2x(x-1) = x(x-3) + 50$

that is, $2x^2 - 2x = x^2 - 3x + 50$

$$x^2 + x = 50$$

Completing the square

$$x^2 + x + \tfrac{1}{4} = 50\tfrac{1}{4}$$
$$(x + \tfrac{1}{2})^2 = \tfrac{201}{4}$$

Taking the square root

$$x + \tfrac{1}{2} = \pm \frac{\sqrt{201}}{2}$$
$$x + \tfrac{1}{2} = \pm \frac{14{\cdot}2}{2}$$

$\therefore\quad x = 6{\cdot}6$ approx. or $-7{\cdot}6$ approx.

Obviously the negative solution is not applicable to the problem.

$\therefore$ the solution is $\boldsymbol{x = 6{\cdot}6}$ **ft.**

Example 3. *If a circular lawn is surrounded by a path of a uniform width of* 3 *ft, and the area of the path is* $\frac{7}{9}$ *that of the lawn, find the radius of the lawn.*

Let R be the radius of the lawn.

Then area of lawn $= \pi R^2$

and area of path and lawn together $= \pi(R + 3)^2$

$\therefore$ Area of path $= \pi(R+3)^2 - \pi R^2$

Hence $\pi(R+3)^2 - \pi R^2 = \frac{7}{9}\pi R^2$

Dividing throughout by π we have:

$$(R+3)^2 - R^2 = \tfrac{7}{9}R^2$$
$$R^2 + 6R + 9 - R^2 = \tfrac{7}{9}R^2$$

Then $\frac{7}{9}R^2 - 6R - 9 = 0$

Multiplying throughout by $\frac{9}{7}$ so that the coefficient of R^2 is unity we get:

$$R^2 - \tfrac{54}{7}R = \tfrac{81}{7}$$
$$R^2 - \tfrac{54}{7}R + (-\tfrac{27}{7})^2 = \tfrac{81}{7} + \tfrac{729}{49}$$
$$(R - \tfrac{27}{7})^2 = \tfrac{1296}{49}$$

Coeff. of R $= -\frac{54}{7}$

$\frac{1}{2}$ Coeff. $= -\frac{27}{7}$

($\frac{1}{2}$ Coeff.)^2 $= \frac{729}{49}$

Then $R - \frac{27}{7} = \pm \frac{36}{7}$

$\therefore\quad R = \frac{27}{7} + \frac{36}{7} = 9$ ft.

or $R = \frac{27}{7} - \frac{36}{7} = -\frac{9}{7}$

The second value is inapplicable in this problem.

$\therefore$ the solution is **R = 9 ft.**

EXERCISE XV

SECTION A

Find the square roots of the following expressions:

1. $x^2 + 4x + 4$.
2. $x^2 - 8x + 16$.
3. $x^2 - x + \frac{1}{4}$.
4. $x^2 - \frac{2}{3}x + \frac{1}{9}$.
5. $x^2 + 40x + 400$.
6. $x^2 - 0{\cdot}2x + 0{\cdot}01$.
7. $R^2 - 5R + 6{\cdot}25$.
8. $\dfrac{1}{x^2} - \dfrac{1}{x} + \frac{1}{4}$.
9. $\dfrac{1}{a^2} - \frac{4}{3}a + \frac{4}{9}$.
10. $4x^2 - 12ax + 9a^2$.
11. $25 - 10x + x^2$.
12. $\dfrac{1}{a^2} + \dfrac{2}{ab} + \dfrac{1}{b^2}$.

SECTION B

Solve the following quadratic equations:

1. $(x-9)^2 = 25$.
2. $(x+4)^2 = 121$.
3. $(x+3)^2 = 7$.
4. $(2x-5)^2 = 25$.
5. $(3x-4)^2 = 81$.
6. $(2x+7)^2 = 11$.
7. $x^2 - 4x + 4 = 25$.
8. $x^2 + 10x + 25 = 49$.

Section C

What must be added to each of the following expressions in order to make a complete square:

1. $x^2 - 5x$.
2. $x^2 + x$.
3. $x^2 + 9x$.
4. $x^2 - 4ax$.
5. $x^2 - 11x$.
6. $x^2 - \frac{5}{2}x$.
7. $x^2 + \frac{1}{3}x$.
8. $\frac{1}{a^2} + \frac{4}{ab}$.
9. $x^2 - 0{\cdot}4x$.
10. $x^2 - 1{\cdot}5x$.

Section D

Find the roots of the following quadratic equations by simplifying where necessary and completing the square:

1. $x^2 + x = 12$.
2. $x^2 = 9x + 22$.
3. $x^2 + 7x - 18 = 0$.
4. $x^2 - 12x = -35$.
5. $x^2 = 5x + 14$.
6. $x^2 + x = 7$.
7. $x^2 - 9x = 22$.
8. $x^2 + 7x = 24$.
9. $x^2 + \frac{1}{5}x = \frac{2}{5}$.
10. $x^2 + \frac{8}{9}x = \frac{4}{9}$.
11. $x^2 - 0{\cdot}4x = 1{\cdot}6$.
12. $x^2 - 1{\cdot}8x = -0{\cdot}24$.
13. $x^2 - 0{\cdot}1x = 0{\cdot}64$.
14. $2x^2 - 3x - 5 = 0$.
15. $3x^2 + 17x = -10$.
16. $6x^2 = 15x + 9$.
17. $2R^2 + 11R = -5$.
18. $3x^2 = 7x + 9$.
19. $2x^2 + 5x = -2$.
20. $2x^2 = -9x + 11$.
21. $3x^2 - 5x = -1$.
22. $2R^2 = 5R + 2$.
23. $5R^2 - 7R = +3$.
24. $x(x - 4) = 5$.
25. $x - 2 = \frac{2}{x}$.
26. $x - \frac{7}{2} + \frac{1}{x} = 0$.
27. $\frac{x - 9}{3} = \frac{x + 5}{x}$.
28. $\frac{1}{x - 1} - \frac{1}{x + 2} = \frac{1}{16}$.
29. Solve for $\frac{1}{R}$ the equation $\frac{1}{R^2} - \frac{5}{R} = 10$.

Section E

Find the roots of the following quadratic equations by employing the factor method:

1. $x^2 - 9x = 36$.
2. $x^2 + 7x + 12 = 0$.
3. $2a^2 - 3a - 5 = 0$.
4. $x^2 = 2x + 99$.
5. $6x^2 + 11x - 35 = 0$.
6. $x^2 - \frac{1}{4}x = \frac{1}{8}$.
7. $x^2 + 0{\cdot}1x - 0{\cdot}02 = 0$.
8. $x^2 = -0{\cdot}5x + 0{\cdot}84$.
9. $9x^2 + 6x - 8 = 0$.
10. $5x^2 = -5x + 10$.

Section F

Form the quadratic equations which have the following pairs of roots:

1. 3 and -2.
2. 5 and 4.
3. $\frac{1}{2}$ and $-\frac{1}{2}$.
4. 2·5 and 1·4.
5. $-0{\cdot}6$ and 1·2.
6. $2a$ and $-a$.

Section G

Miscellaneous Exercises and Problems

1. (i) Solve the equation $12x^2 - 26x + 12 = 0$.

(ii) The resistance R lb wt offered to the motion of a motor car when travelling at V m.p.h. is given by $R = A + \frac{V^2}{B}$, where A and B are constants. If $R = 8$ when $V = 30$ and $R = 12$ when $V = 40$, find the values of A and B and find R when $V = 60$ m.p.h.

(Sunderland.)

2. Solve by completing the square, the equation

$$3x^2 + 11x - 42 = 0. \qquad \text{(U.L.C.I.)}$$

3. A train travels a certain distance S miles at a uniform speed of V m.p.h. If the speed were 9 m.p.h. more the journey would take 3 hours less; if the speed were 6 m.p.h.

less, the time taken would be 3 hours more. Find the distance S. (Coventry.)

4. (*a*) Solve:

(i) $\frac{x-5}{3} + \frac{2x-5}{2} = \frac{20x-5}{30}$;

(ii) $x + 3y + 3 = 0$; $3x = 7 - y$.

(*b*) A man walks a distance of 8 miles at a certain speed. He cycles back 6 m.p.h. faster than walking and takes one third of the time. Find his walking speed. (E.M.E.U.)

5. (*a*) Solve $6x^2 = 91 - 5x$.

(*b*) A man motors 72 miles at a certain speed. If he had travelled 6 m.p.h. slower his journey would have taken him 1 hour longer. Find his original speed. (Rugby.)

6. (*a*) Solve: (i) $2x^2 - 7x + 4 = 0$;

(ii) $x^2 = 3 - 4x$.

(*b*) The diagonals of a rectangle are each 20 ft, and the length of the rectangle is twice the breadth; find the dimensions of the rectangle. (Rugby.)

7. Plot the graph of $y = 3x^2 - 28x + 10$ for values of x from 0 to 10. Then:

(*a*) From the graph read: (i) the minimum value for y; (ii) the values of x which satisfy the equation $3x^2 - 28x + 10 = 0$.

(*b*) Solve the equation $3x^2 - 28x + 10 = 0$ by using the formula. (Coventry.)

8. The area of a rectangle of length 8 in. and breadth 5 in. is unchanged if the length is increased by $4x$ in. and the breadth reduced by x in. Form an equation in x and solve it. (U.L.C.I.)

9. The bending moment for a certain uniform beam is given by $M = 25x - \frac{Wx^2}{2}$, where M is the bending moment at a distance x ft from one end, and W is the weight per foot of the beam. Find how far from one end the bending moment has a value of 60 if the weight per foot of the beam is 5 lb. (Dudley.)

10. The diagonal of a rectangle is 1·7 in. long and one side is 0·4 in. longer than the other. Find the lengths of the sides to the nearest hundredth of an inch.

11. A lawn is 14 yd long and 10 yd wide. Round the lawn there is a gravel walk. The area of the lawn is $\frac{7}{9}$ that of the gravel walk. Find the width of the walk.

12. A certain quantity R when multiplied by $2R - 1$ gives 6 as a result. What is the quantity?

13. A diameter of a circle bisects a chord at right angles. If the diameter be 12 in. long and the chord is 10 in. long, find the heights of the segments.

14. The total surface of a cylinder is 24π sq in. If the height be 4 in., what is the radius of the cross-section?

15. A body travels at x ft per sec for 10 sec, and afterwards for another $4x$ sec at the same rate. If the total distance is 126 ft, what is the value of x?

16. The relation between the joint resistance R and two resistances r_1 and r_2 in parallel, is given by the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

If $R = 12$ ohms, and r_2 is 6 ohms greater than r_1, find r_1 and r_2.

17. The strength of a beam depends upon its material and its section modulus, Z. For a rectangular beam $Z = \frac{bd^2}{6}$, where b and d are the breadth and depth respectively, usually measured in inches.

For 3-in.-wide timber joists determine the depth to give a strength twice as great as that of a 5-in-deep joist.

18. The volume V of the frustum of a cone is given by the formula $V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$. Find r is $\pi = \frac{22}{7}$, $R = 5$ in., $h = 6$ in., and $V = 308$ cu in.

19. The area of a rectangle is 12 sq in. and its perimeter is 16 in. Find the lengths of its sides.

20. The formula giving the sag D in a cable of length L and span S is expressed by $L = \frac{8D^2}{3S} + S$. Find S when $L = 88{\cdot}4$, $D = 2{\cdot}4$.

21. The slant side of a cone is 15 in. long and the height is 3 in. longer than the radius of the base. Find the height and radius of the base.

CHAPTER 16

VECTORS

1. The quantities which have been dealt with so far in this book are subject to the usual operations of arithmetic, and any one of them can be expressed by a simple arithmetical number.

Thus a length, an area, a mass, a weight or a volume is usually expressed as a mere number in terms of its own particular unit.

Such quantities are called **scalar** quantities.

Other quantities, however, such as a **displacement,** a **force,** a **velocity, momentum,** etc., cannot be fully expressed by a mere number, as each of them involves **direction** as well as **magnitude.**

For example, the motion of an aeroplane is not fully defined by the statement that it is moving at 156 m.p.h. Its **direction** must be given as well.

Again, if it be stated in a police court that a motor involved in an accident was moving at 35 m.p.h., this is not complete evidence. It is necessary also to know in which **direction** it was moving.

Thus a velocity is not completely defined unless we state both its **magnitude and its direction.**

Quantities, such as velocities, which involve both direction and magnitude are called **Vector Quantities.**

2. Linear Displacements

If a particle moves from A to B a distance of 3 in., and then from B to C a further distance of 2 in. (Fig. 128(*a*)) in

the same straight line and in the same direction, its net displacement is

$$AB + BC = 3 + 2 = 5 \text{ in.} = AC$$

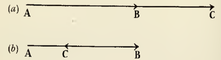


FIG. 128.

If, however, after moving to B, it reverses its direction, the net displacement is $AB - BC = AC = 1$ in. (Fig. 128(*b*)).

In these two cases the net displacement has been found arithmetically, or algebraically.

3. Displacements not in the Same Straight Line

Now suppose the particle to move from A to B as before, and then to change its direction and move to C so that BC makes an angle of 45° with the original direction (Fig. 129).

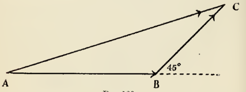


FIG. 129.

As before, AC still represents the **net** or **resultant displacement,** where C has been reached by two steps, but AC is not in this case the arithmetical sum of the values of AB and BC.

In other words, AB + BC treated as an arithmetical or algebraical expression **does not give AC as a resultant.**

The displacement represented by AC is equivalent to the displacement represented by AB and BC, and the latter are called the **component displacements of AC.**

4. Representation of Velocities

Velocities involve **magnitude** and **direction,** and hence the method of representing them is similar to that employed in the case of displacements. They also are **vector** quantities.

Example. *The air-speed of an aeroplane is 120 m.p.h. If it steers in a north-easterly direction when a west wind is blowing at 40 m.p.h., what will be its actual path in the air, and how far will it be from its starting point at the end of 1 hour?*

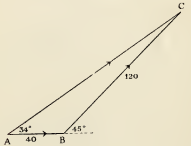


FIG. 130.

The resultant path of the aeroplane is due to the combined effect of the wind and its own air-speed.

The wind carries the aeroplane 40 miles due east in 1 hour. Hence draw AB (Fig. 130) from west to east to represent 40 miles. At the same time the aeroplane, by its

own movement, would travel 120 miles in a north-easterly direction.

From B draw BC to represent 120 miles to the north-east.

Join AC.

Then C is the point reached in I hour, and the actual path of the aeroplane is along the line AC.

Hence AC represents the **resultant** velocity of the aeroplane. It also represents the distance from the starting point after 1 hour.

AC is a **vector** quantity and represents the **vector** sum of the two **vector** quantities AB and BC. Measured to the same scale as AB and BC, AC = 151 m.p.h., and its direction is 34° north of east.

5. Representation of Forces

Since **Forces** also involve magnitude and direction in their representation, they are **vector** quantities, and can in general be treated as in the two previous cases.

Example. *Two forces of* 1·8 *lb wt and* 1·2 *lb wt act at a point in a body at right angles. Find a force which is equivalent to them.*

We have to find the net result of these two forces—in other words, we have to find their vector sum.

Since their directions are not given with reference to any set direction, we will take the 1·8 lb as acting from west to east and the 1·2 lb as acting from south to north.

Hence draw AB to represent 1·8 lb wt, then to the same scale draw BC at right angles to AB to represent 1·2 lb wt.

Join AC.

Then AC is the **vector sum** of AB and BC, and as such represents a **force** which is equivalent to the two given forces.

Measured to scale or by calculation AC = a force of 2·16 lb wt, and its direction is along AC, which makes an angle of $33\frac{1}{2}°$ with AB.

6. In order to avoid all ambiguity with regard to the representation of a **vector** quantity, we must have a standard or basic direction so that it can be completely specified.

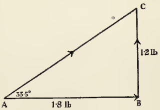


FIG. 131.

The cases dealt with show that we must know

(1) Its **direction** in relation to some **fixed** or **standard** direction.

In the second case, which dealt with velocities, the directions were given in relation to an east and west line.

(2) Its **magnitude,** which is shown by the length of the line drawn to a chosen scale.

(3) Its **sense**—that is, the movement along the line itself indicated by an arrow.

[Students of Mechanics will realise that an additional property must be given before a **force** can be completely known. This third property is its "line of action" or "point of application." Though this third property is of importance, it is generally sufficient simply to regard a force as a vector quantity.]

7. Specifications of Vectors

Let OX (Fig. 132), drawn from left to right, be the axis of reference or the standard direction to which the direction of any vector can be referred.

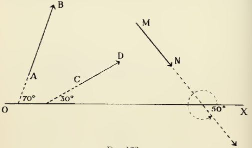


FIG. 132.

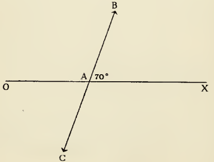


FIG. 133.

(1) Suppose a point to be displaced from A to B a distance of 4 units, so that AB makes an angle of 70° with the +ve direction of the axis OX. Then as a vector we represent AB by $4_{70°}$.

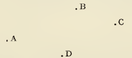
(2) If CD is 2·5 units it represents the **vector** $2{\cdot}5_{30°}$.

(3) MN, however, is a line which makes an angle of 310° with the +ve direction of OX, since a rotation of 310° is required to bring it from the fixed direction OX to its present direction. If it is 3 units in length, it represents the vector $3_{310°}$.

(4) The vector $-\,3_{70°}$ indicates a **reversal** in direction. It has the same numerical magnitude as + 3, but it is in the opposite direction. It is therefore the same as $3_{250°}$.

Thus AB ≡ the vector $3_{70°}$ (Fig. 133)
and AC ≡ the vector $-\,3_{70°}$ or $3_{250°}$.

(5) Let A, B, C, D, etc., be points lying in the same plane.



Then a displacement from A to B is sometimes indicated by $\overline{AB}$, a displacement from D to B by $\overline{DB}$, and so on.

Hence under these conditions $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 0$, since as a result of the four displacements, the starting point and the finishing point are identical. In other words, the sum of the **vectors** is zero.

8. Sum of Two or More Vectors

In the section on **displacement** we found that AC was the **resultant displacement** of AB and BC, and since each of these is a vector, we can say that AC represents the **vector sum** of AB and AC.

To illustrate this further, find the sum of the vectors $+3_{0°}, +2_{45°}, +3_{90°}$.

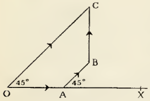


FIG. 134.

Let OX represent the basic direction.

Set off OA along OX equal to 3 units. This represents the vector $3_{0°}$.

Draw AB, making an angle of 45° with OX and 2 units in length.

Then AB ≡ the vector $2_{45°}$.

Draw BC at right angles to OX and 3 units in length.

Then BC ≡ the vector $3_{90°}$.

Then OC represents the sum of the vectors $3_{0°}, 2_{45°}, 3_{90°}$.

By measurement OC = 6·24 units in length and OC makes an angle of 45° with OX. Then OC ≡ the resultant vector $6{\cdot}24_{45°}$.

$$\therefore \quad +3_{0°}+2_{45°}+3_{90°}=6{\cdot}24_{45°}$$

As a point of interest, if the student refers to Fig. 123, he will see that in determining the resultant velocity we found that

$$40_{0°}+120_{45°}=151_{34°}$$

9. Given the Vector Sum of Two Quantities and One of its Components, to Find the Other Component

Example. *Let* $3{\cdot}5_{60}$ ≡ *the vector sum, and* $2{\cdot}8_{30}$ *one of its components. Find the other component.*

Let OX represent the basic or standard line.

Draw MN, making an angle of 60° with the positive direction of OX and 3·5 units in length.

Draw MK to represent $2{\cdot}8_{30°}$.

Join NK.

Then KN ≡ the other component vector.

Produce NK to meet OX in S.

Measure the angle XSN.

Length of NK = 1·75 units. ∠XSN = 112° approx.

Hence $1{\cdot}75_{112°}$ ≡ the other component vector.

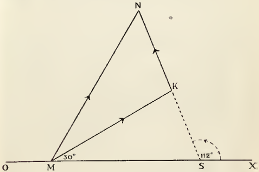


FIG. 135.

EXERCISE XVI

Find the following vector sums:

1. $4_{35°}+5_{125°}$.
2. $2_{30°}+3_{50°}+4_{60°}$.
3. $3{\cdot}5_{120°}+2{\cdot}5_{30°}+4_{60°}+3_{90°}$.
4. If $a=4_{0°}$, $b=3{\cdot}5_{30°}$, $c=3_{60°}$, find the value of $a+b-c$, and of $a-b+c$.
5. A ship can steam in still water at 20 m.p.h. If it steers due north, and is subjected to a tide running due east at 4 m.p.h., and to a wind which by itself would

cause it to move in a south-westerly direction at 5 m.p.h., what course does the ship take, and what would be its velocity along that course?

6. Find the resultant of the following forces:

12 lb acting at an angle of 25° with the standard line OX.
8 lb acting at an angle of 130° with OX.
6 lb acting at an angle of 250° with OX.

7. A ship starts from a point O and travels at 15 m.p.h. in a direction 60° south of west for 2 hours. How far south, and how far west, of O will it be at the end of this time? (Rugby.)

8. (i) Plot the vectors $2_{35°}$ and $33_{52°}$. Find (*a*) the resultant of these two vectors, and (*b*) the horizontal and vertical components of the resultant.

(ii) A ship can steam at 21 knots in still water. If its course is N. 40° E., and a current is flowing towards the south-east at 5 knots, find the actual speed and course of the ship. (Sunderland.)

9. (i) Find graphically, or by calculation, the resultant of the vectors $3_{25°}$ and $5_{125°}$.

(ii) A flag-pole is held vertical by two ropes in the same vertical plane, making angles of 30° and 65° with it on opposite sides. If the tensions in the ropes are 12 lb and 10 lb respectively find the resultant pull on the flag-pole. (Sunderland.)

10. Represent by means of a diagram four forces all pulling away from a point. The first is 4 lb acting east, the second 4 lb acting 60° north of east, the third 1 lb acting 60° north of west and the fourth 3 lb acting west.

Then graphically or otherwise find a single force which will balance all the above forces. (U.E.I.)

11. Find the vector equivalent to

$$5_{0°} + 4_{120°} - 3_{60°}$$

12. If A, B, C and D are any four points, prove that $\overline{AB} + \overline{CD} = \overline{AD} + \overline{CB}$. (N.C.T.E.C.)

13. A body undergoes successive displacements whose vectors are $3_{45°}$ and $5_{30°}$. Find graphically or otherwise the vector of the single displacement which would cause the same total change in the position of the body. (N.C.T.E.C.)

14. A point on the connecting-rod of an engine is moving forward horizontally at 5 ft per sec. At the same instant this point has a velocity of 3 ft per sec in the same vertical plane, but inclined at 120° to the direction of horizontal motion. By means of a scale drawing, represent these velocities and find the magnitude and direction of the actual velocity of the point. (U.E.I.)

15. A body undergoes a displacement of 4 in. in a north-east direction and then 3 in. in a direction 30° N. of E.

Find by means of a diagram, the magnitude and direction of a single displacement which would cause the same change in the position of the body. (U.E.I.)

CONSTANTS

Constant.	Number.	Log.
π	3·1416	0·49715
$\frac{\pi}{4}$	0·7854	$\bar{1}$·89509
$\frac{1}{\pi}$	0·3183	$\bar{1}$·50285
π^2	9·8696	0·99430
$\sqrt{\pi}$	1·7725	0·24857
$\frac{4}{3}\pi$	4·1888	0·62209
$\frac{180}{\pi}$	57·2958	1·75812
$\frac{\pi}{180}$	0·01745	$\bar{2}$·24188
e	2·71828	0·43429
$\log_e 10$	2·3026	0·36222

CONVERSION FACTORS

To convert	Multiply by	Log.
Metres to inches	39·37	1·59517
Inches to centimetres	2·5400	0·40483
Kilometres to miles	0·6214	$\bar{1}$·79335
Kilograms to pounds	2·20462	0·34333
Pounds to kilograms	0·45359	$\bar{1}$·65666
Gallons to cubic inches	277·45	2·44318
Radians to degrees	57·2958	1·75812
Miles per hour to feet per second	1·4666	0·1663

g. (at Greenwich) = 32·191 ft per sec^2
= 981·18 cm per sec^2
Weight of 1 cu ft of water = 62·42 lb (at 4° C.)

No.	Log.	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1·1	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1·2	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1·3	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1·4	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1·5	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1·6	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1·7	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1·8	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1·9	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2·0	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2·1	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2·2	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2·3	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2·4	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2·5	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2·6	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
2·7	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
2·8	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
2·9	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
3·0	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
3·1	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
3·2	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
3·3	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
3·4	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
3·5	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
3·6	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
3·7	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
3·8	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
3·9	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
4·0	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
4·1	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
4·2	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
4·3	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
4·4	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
4·5	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
4·6	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
4·7	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
4·8	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
4·9	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
5·0	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
5·1	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
5·2	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
5·3	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
5·4	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

No.	Log.	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5·5	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
5·6	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
5·7	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
5·8	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
5·9	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
6·0	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
6·1	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
6·2	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
6·3	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
6·4	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
6·5	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
6·6	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
6·7	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
6·8	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
6·9	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
7·0	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
7·1	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
7·2	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
7·3	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
7·4	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
7·5	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
7·6	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
7·7	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
7·8	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
7·9	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
8·0	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
8·1	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
8·2	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
8·3	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
8·4	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
8·5	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
8·6	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
8·7	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
8·8	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
8·9	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
9·0	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
9·1	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
9·2	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
9·3	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
9·4	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
9·5	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
9·6	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
9·7	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
9·8	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
9·9	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	2
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	3
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	3	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	·0349	·0366	·0384	·0401	·0419	·0436	·0454	·0471	·0488	·0506	3	6	9	12	15
3°	·0523	·0541	·0558	·0576	·0593	·0610	·0628	·0645	·0663	·0680	3	6	9	12	15
4°	·0698	·0715	·0732	·0750	·0767	·0785	·0802	·0819	·0837	·0854	3	6	9	12	14
5°	·0872	·0889	·0906	·0924	·0941	·0958	·0976	·0993	·1011	·1028	3	6	9	12	14
6°	·1045	·1063	·1080	·1097	·1115	·1132	·1149	·1167	·1184	·1201	3	6	9	12	14
7°	·1219	·1236	·1253	·1271	·1288	·1305	·1323	·1340	·1357	·1374	3	6	9	12	14
8°	·1392	·1409	·1426	·1444	·1461	·1478	·1495	·1513	·1530	·1547	3	6	9	12	14
9°	·1564	·1582	·1599	·1616	·1633	·1650	·1668	·1685	·1702	·1719	3	6	9	12	14
10°	·1736	·1754	·1771	·1788	·1805	·1822	·1840	·1857	·1874	·1891	3	6	9	11	14
11°	·1908	·1925	·1942	·1959	·1977	·1994	·2011	·2028	·2045	·2062	3	6	9	11	14
12°	·2079	·2096	·2113	·2130	·2147	·2164	·2181	·2198	·2215	·2233	3	6	9	11	14
13°	·2250	·2267	·2284	·2300	·2317	·2334	·2351	·2368	·2385	·2402	3	6	8	11	14
14°	·2419	·2436	·2453	·2470	·2487	·2504	·2521	·2538	·2554	·2571	3	6	8	11	14
15°	·2588	·2605	·2622	·2639	·2656	·2672	·2689	·2706	·2723	·2740	3	6	8	11	14
16°	·2756	·2773	·2790	·2807	·2823	·2840	·2857	·2874	·2890	·2907	3	6	8	11	14
17°	·2924	·2940	·2957	·2974	·2990	·3007	·3024	·3040	·3057	·3074	3	6	8	11	14
18°	·3090	·3107	·3123	·3140	·3156	·3173	·3190	·3206	·3223	·3239	3	6	8	11	14
19°	·3256	·3272	·3289	·3305	·3322	·3338	·3355	·3371	·3387	·3404	3	5	8	11	14
20°	·3420	·3437	·3453	·3469	·3486	·3502	·3518	·3535	·3551	·3567	3	5	8	11	14
21°	·3584	·3600	·3616	·3633	·3649	·3665	·3681	·3697	·3714	·3730	3	5	8	11	14
22°	·3746	·3762	·3778	·3795	·3811	·3827	·3843	·3859	·3875	·3891	3	5	8	11	14
23°	·3907	·3923	·3939	·3955	·3971	·3987	·4003	·4019	·4035	·4051	3	5	8	11	14
24°	·4067	·4083	·4099	·4115	·4131	·4147	·4163	·4179	·4195	·4210	3	5	8	11	13
25°	·4226	·4242	·4258	·4274	·4289	·4305	·4321	·4337	·4352	·4368	3	5	8	11	13
26°	·4384	·4399	·4415	·4431	·4446	·4462	·4478	·4493	·4509	·4524	3	5	8	10	13
27°	·4540	·4555	·4571	·4586	·4602	·4617	·4633	·4648	·4664	·4679	3	5	8	10	13
28°	·4695	·4710	·4726	·4741	·4756	·4772	·4787	·4802	·4818	·4833	3	5	8	10	13
29°	·4848	·4863	·4879	·4894	·4909	·4924	·4939	·4955	·4970	·4985	3	5	8	10	13
30°	·5000	·5015	·5030	·5045	·5060	·5075	·5090	·5105	·5120	·5135	3	5	8	10	13
31°	·5150	·5165	·5180	·5195	·5210	·5225	·5240	·5255	·5270	·5284	2	5	7	10	12
32°	·5299	·5314	·5329	·5344	·5358	·5373	·5388	·5402	·5417	·5432	2	5	7	10	12
33°	·5446	·5461	·5476	·5490	·5505	·5519	·5534	·5548	·5563	·5577	2	5	7	10	12
34°	·5592	·5606	·5621	·5635	·5650	·5664	·5678	·5693	·5707	·5721	2	5	7	10	12
35°	·5736	·5750	·5764	·5779	·5793	·5807	·5821	·5835	·5850	·5864	2	5	7	9	12
36°	·5878	·5892	·5906	·5920	·5934	·5948	·5962	·5976	·5990	·6004	2	5	7	9	12
37°	·6018	·6032	·6046	·6060	·6074	·6088	·6101	·6115	·6129	·6143	2	5	7	9	12
38°	·6157	·6170	·6184	·6198	·6211	·6225	·6239	·6252	·6266	·6280	2	5	7	9	11
39°	·6293	·6307	·6320	·6334	·6347	·6361	·6374	·6388	·6401	·6414	2	4	7	9	11
40°	·6428	·6441	·6455	·6468	·6481	·6494	·6508	·6521	·6534	·6547	2	4	7	9	11
41°	·6561	·6574	·6587	·6600	·6613	·6626	·6639	·6652	·6665	·6678	2	4	7	9	11
42°	·6691	·6704	·6717	·6730	·6743	·6756	·6769	·6782	·6794	·6807	2	4	6	9	11
43°	·6820	·6833	·6845	·6858	·6871	·6884	·6896	·6909	·6921	·6934	2	4	6	8	11
44°	·6947	·6959	·6972	·6984	·6997	·7009	·7022	·7034	·7046	·7059	2	4	6	8	10

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4	5′
45°	·7071	·7083	·7096	·7108	·7120	·7133	·7145	·7157	·7169	·7181	2	4	6	8	10
46°	·7193	·7206	·7218	·7230	·7242	·7254	·7266	·7278	·7290	·7302	2	4	6	8	10
47°	·7314	·7325	·7337	·7349	·7361	·7373	·7385	·7396	·7408	·7420	2	4	6	8	10
48°	·7431	·7443	·7455	·7466	·7478	·7490	·7501	·7513	·7524	·7536	2	4	6	8	10
49°	·7547	·7559	·7570	·7581	·7593	·7604	·7615	·7627	·7638	·7649	2	4	6	8	9
50°	·7660	·7672	·7683	·7694	·7705	·7716	·7727	·7738	·7749	·7760	2	4	6	7	9
51°	·7771	·7782	·7793	·7804	·7815	·7826	·7837	·7848	·7859	·7869	2	4	5	7	9
52°	·7880	·7891	·7902	·7912	·7923	·7934	·7944	·7955	·7965	·7976	2	4	5	7	9
53°	·7986	·7997	·8007	·8018	·8028	·8039	·8049	·8059	·8070	·8080	2	3	5	7	9
54°	·8090	·8100	·8111	·8121	·8131	·8141	·8151	·8161	·8171	·8181	2	3	5	7	8
55°	·8192	·8202	·8211	·8221	·8231	·8241	·8251	·8261	·8271	·8281	2	3	5	7	8
56°	·8290	·8300	·8310	·8320	·8329	·8339	·8348	·8358	·8368	·8377	2	3	5	6	8
57°	·8387	·8396	·8406	·8415	·8425	·8434	·8443	·8453	·8462	·8471	2	3	5	6	8
58°	·8480	·8490	·8499	·8508	·8517	·8526	·8536	·8545	·8554	·8563	2	3	5	6	8
59°	·8572	·8581	·8590	·8599	·8607	·8616	·8625	·8634	·8643	·8652	1	3	4	6	7
60°	·8660	·8669	·8678	·8686	·8695	·8704	·8712	·8721	·8729	·8738	1	3	4	6	7
61°	·8746	·8755	·8763	·8771	·8780	·8788	·8796	·8805	·8813	·8821	1	3	4	6	7
62°	·8829	·8838	·8846	·8854	·8862	·8870	·8878	·8886	·8894	·8902	1	3	4	5	7
63°	·8910	·8918	·8926	·8934	·8942	·8949	·8957	·8965	·8973	·8980	1	3	4	5	6
64°	·8988	·8996	·9003	·9011	·9018	·9026	·9033	·9041	·9048	·9056	1	3	4	5	6
65°	·9063	·9070	·9078	·9085	·9092	·9100	·9107	·9114	·9121	·9128	1	2	4	5	6
66°	·9135	·9143	·9150	·9157	·9164	·9171	·9178	·9184	·9191	·9198	1	2	3	5	6
67°	·9205	·9212	·9219	·9225	·9232	·9239	·9245	·9252	·9259	·9265	1	2	3	4	6
68°	·9272	·9278	·9285	·9291	·9298	·9304	·9311	·9317	·9323	·9330	1	2	3	4	5
69°	·9336	·9342	·9348	·9354	·9361	·9367	·9373	·9379	·9385	·9391	1	2	3	4	5
70°	·9397	·9403	·9409	·9415	·9421	·9426	·9432	·9438	·9444	·9449	1	2	3	4	5
71°	·9455	·9461	·9466	·9472	·9478	·9483	·9489	·9494	·9500	·9505	1	2	3	4	5
72°	·9511	·9516	·9521	·9527	·9532	·9537	·9542	·9548	·9553	·9558	1	2	3	3	4
73°	·9563	·9568	·9573	·9578	·9583	·9588	·9593	·9598	·9603	·9608	1	2	2	3	4
74°	·9613	·9617	·9622	·9627	·9632	·9636	·9641	·9646	·9650	·9655	1	2	2	3	4
75°	·9659	·9664	·9668	·9673	·9677	·9681	·9686	·9690	·9694	·9699	1	1	2	3	4
76°	·9703	·9707	·9711	·9715	·9720	·9724	·9728	·9732	·9736	·9740	1	1	2	3	3
77°	·9744	·9748	·9751	·9755	·9759	·9763	·9767	·9770	·9774	·9778	1	1	2	3	3
78°	·9781	·9785	·9789	·9792	·9796	·9799	·9803	·9806	·9810	·9813	1	1	2	2	3
79°	·9816	·9820	·9823	·9826	·9829	·9833	·9836	·9839	·9842	·9845	1	1	2	2	3
80°	·9848	·9851	·9854	·9857	·9860	·9863	·9866	·9869	·9871	·9874	0	1	1	2	2
81°	·9877	·9880	·9882	·9885	·9888	·9890	·9893	·9895	·9898	·9900	0	1	1	2	2
82°	·9903	·9905	·9907	·9910	·9912	·9914	·9917	·9919	·9921	·9923	0	1	1	2	2
83°	·9925	·9928	·9930	·9932	·9934	·9936	·9938	·9940	·9942	·9943	0	1	1	1	2
84°	·9945	·9947	·9949	·9951	·9952	·9954	·9956	·9957	·9959	·9960	0	1	1	1	1
85°	·9962	·9963	·9965	·9966	·9968	·9969	·9971	·9972	·9973	·9974	0	0	1	1	1
86°	·9976	·9977	·9978	·9979	·9980	·9981	·9982	·9983	·9984	·9985	0	0	1	1	1
87°	·9986	·9987	·9988	·9989	·9990	·9990	·9991	·9992	·9993	·9993	0	0	0	1	1
88°	·9994	·9995	·9995	·9996	·9996	·9997	·9997	·9997	·9998	·9998	0	0	0	0	0
89°	·9998	·9999	·9999	·9999	·9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	·9999	·9999	·9999	·9999	0	0	0	0	0
1°	·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995	0	0	0	0	0
2°	·9994	·9993	·9993	·9992	·9991	·9990	·9990	·9989	·9988	·9987	0	0	0	0	0
3°	·9986	·9985	·9984	·9983	·9982	·9981	·9980	·9979	·9978	·9977	0	0	1	1	1
4°	·9976	·9974	·9973	·9972	·9971	·9969	·9968	·9966	·9965	·9963	0	0	1	1	1
5°	·9962	·9960	·9959	·9957	·9956	·9954	·9952	·9951	·9949	.9947	0	1	1	1	1
6°	·9945	·9943	·9942	·9940	·9938	·9936	·9934	·9932	·9930	·9928	0	1	1	1	2
7°	·9925	·9923	·9921	·9919	·9917	·9914	·9912	·9910	·9907	·9905	0	1	1	2	2
8°	·9903	·9900	·9898	·9895	·9893	·9890	·9888	·9885	·9882	·9880	0	1	1	2	2
9°	·9877	·9874	·9871	·9869	·9866	·9863	·9860	·9857	·9854	·9851	0	1	1	2	2
10°	·9848	·9845	·9842	·9839	·9836	·9833	·9829	·9826	·9823	·9820	1	1	2	2	3
11°	·9816	·9813	·9810	·9806	·9803	·9799	·9796	·9792	·9789	·9785	1	1	2	2	3
12°	·9781	·9778	·9774	·9770	·9767	·9763	·9759	·9755	·9751	·9748	1	1	2	3	3
13°	·9744	·9740	·9736	·9732	·9728	·9724	·9720	·9715	·9711	·9707	1	1	2	3	3
14°	·9703	·9699	·9694	·9690	·9686	·9681	·9677	·9673	·9668	·9664	1	1	2	3	4
15°	·9659	·9655	·9650	·9646	·9641	·9636	·9632	·9627	·9622	·9617	1	2	2	3	4
16°	·9613	·9608	·9003	·9598	·9593	·9588	·9583	·9578	·9573	·9568	1	2	2	3	4
17°	·9563	·9558	·9553	·9548	·9542	·9537	·9532	·9527	·9521	·9516	1	2	3	4	4
18°	·9511	·9505	·9500	·9494	·9489	·9483	·9478	·9472	·9466	·9461	1	2	3	4	5
19°	·9455	·9449	·9444	·9438	·9432	·9426	·9421	·9415	·9409	·9403	1	2	3	4	5
20°	·9397	·9391	·9385	·9379	·9373	·9367	·9361	·9354	·9348	·9342	1	2	3	4	5
21°	·9336	·9330	·9323	·9317	·9311	·9304	·9298	·9291	·9285	·9278	1	2	3	4	5
22°	·9272	·9265	·9259	·9252	·9245	·9239	·9232	·9225	·9219	·9212	1	2	3	4	6
23°	·9205	·9198	·9191	·9184	·9178	·9171	·9164	·9157	·9150	·9143	1	2	3	5	6
24°	·9135	·9128	·9121	·9114	·9107	·9100	·9092	·9085	·9078	·9070	1	2	4	5	6
25°	·9063	·9056	·9048	·9041	·9033	·9026	·9018	·9011	·9003	·8996	1	3	4	5	6
26°	·8988	·8980	·8973	·8965	·8957	·8949	·8942	·8934	·8926	·8918	1	3	4	5	6
27°	·8910	·8902	·8894	·8886	·8878	·8870	·8862	·8854	·8846	·8838	1	3	4	5	7
28°	·8829	·8821	·8813	·8805	·8796	·8788	·8780	·8771	·8763	·8755	1	3	4	6	7
29°	·8746	·8738	·8729	·8721	·8712	·8704	·8695	·8686	·8678	·8669	1	3	4	6	7
30°	·8660	·8652	·8643	·8634	·8625	·8616	·8607	·8599	·8590	·8581	1	3	4	6	7
31°	·8572	·8563	·8554	·8545	·8536	·8526	·8517	·8508	·8499	·8490	2	3	5	6	8
32°	·8480	·8471	·8462	·8453	·8443	·8434	·8425	·8415	·8406	·8396	2	3	5	6	8
33°	·8387	·8377	·8368	·8358	·8348	·8339	·8329	·8320	·8310	·8300	2	3	5	6	8
34°	·8290	·8281	·8271	·8261	·8251	·8241	·8231	·8221	·8211	·8202	2	3	5	7	8
35°	·8192	·8181	·8171	·8161	·8151	·8141	·8131	·8121	·8111	·8100	2	3	5	7	8
36°	·8090	·8080	·8070	·8059	·8049	·8039	·8028	·8018	·8007	·7997	2	3	5	7	9
37°	·7986	·7976	·7965	·7955	·7944	·7934	·7923	·7912	·7902	·7891	2	4	5	7	9
38°	·7880	·7869	·7859	·7848	·7837	·7826	·7815	·7804	·7793	·7782	2	4	5	7	9
39°	·7771	·7760	·7749	·7738	·7727	·7716	·7705	·7694	·7683	·7672	2	4	6	7	9
40°	·7660	·7649	·7638	·7627	·7615	·7604	·7593	·7581	·7570	·7559	2	4	6	8	9
41°	·7547	·7536	·7524	·7513	·7501	·7490	·7478	·7466	·7455	·7443	2	4	6	8	10
42°	·7431	·7420	·7408	·7396	·7385	·7373	·7361	·7349	·7337	·7325	2	4	6	8	10
43°	·7314	·7302	·7290	·7278	·7266	·7254	·7242	·7230	·7218	·7206	2	4	6	8	10
44°	·7193	·7181	·7169	·7157	·7145	·7133	·7120	·7108	·7096	·7083	2	4	6	8	10

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5
45°	·7071	·7059	·7046	·7034	·7022	·7009	·6997	·6984	·6972	·6959	2	4	6	8	10
46°	·6947	·6934	·6921	·6909	·6896	·6884	·6871	·6858	·6845	·6833	2	4	6	8	11
47°	·6820	·6807	·6794	·6782	·6769	·6756	·6743	·6730	·6717	·6704	2	4	6	9	11
48°	·6691	·6678	·6665	·6652	·6639	·6626	·6613	·6600	·6587	·6574	2	4	7	9	11
49°	·6561	·6547	·6534	·6521	·6508	·6494	·6481	·6468	·6455	·6441	2	4	7	9	11
50°	·6428	·6414	·6401	·6388	·6374	·6361	·6347	·6334	·6320	·6307	2	4	7	9	11
51°	·6293	·6280	·6266	·6252	·6239	·6225	·6211	·6198	·6184	·6170	2	5	7	9	11
52°	·6157	·6143	·6129	·6115	·6101	·6088	·6074	·6060	·6046	·6032	2	5	7	9	12
53°	·6018	·6004	·5990	·5976	·5962	·5948	·5934	·5920	·5906	·5892	2	5	7	9	12
54°	·5878	·5864	·5850	·5835	·5821	·5807	·5793	·5779	·5764	·5750	2	5	7	9	12
55°	·5736	·5721	·5707	·5693	·5678	·5664	·5650	·5635	·5621	·5606	2	5	7	10	12
56°	·5592	·5577	·5563	·5548	·5534	·5519	·5505	·5490	·5476	·5461	2	5	7	10	12
57°	·5446	·5432	·5417	·5402	·5388	·5373	·5358	·5344	·5329	·5314	2	5	7	10	12
58°	·5299	·5284	·5270	·5255	·5240	·5225	·5210	·5195	·5180	·5165	2	5	7	10	12
59°	·5150	·5135	·5120	·5105	·5090	·5075	·5060	·5045	·5030	·5015	3	5	8	10	13
60°	·5000	·4985	·4970	·4955	·4939	·4924	·4909	·4894	·4879	·4863	3	5	8	10	13
61°	·4848	·4833	·4818	·4802	·4787	·4772	·4756	·4741	·4726	·4710	3	5	8	10	13
62°	·4695	·4679	·4664	·4648	·4633	·4617	·4602	·4586	·4571	·4555	3	5	8	10	13
63°	·4540	·4524	·4509	·4493	·4478	·4462	·4446	·4431	·4415	·4399	3	5	8	10	13
64°	·4384	·4368	·4352	·4337	·4321	·4305	·4289	·4274	·4258	·4242	3	5	8	11	13
65°	·4226	·4210	·4195	·4179	·4163	·4147	·4131	·4115	·4099	·4083	3	5	8	11	13
66°	·4067	·4051	·4035	·4019	·4003	·3987	·3971	·3955	·3939	·3923	3	5	8	11	13
67°	·3907	·3891	·3875	·3859	·3843	·3827	·3811	·3795	·3778	·3762	3	5	8	11	13
68°	·3746	·3730	·3714	·3697	·3681	·3665	·3649	·3633	·3616	·3600	3	5	8	11	14
69°	·3584	·3567	·3551	·3535	·3518	·3502	·3486	·3469	·3453	·3437	3	5	8	11	14
70°	·3420	·3404	·3387	·3371	·3355	·3338	·3322	·3305	·3289	·3272	3	5	8	11	14
71°	·3256	·3239	·3223	·3206	·3190	·3173	·3156	·3140	·3123	·3107	3	6	8	11	14
72°	·3090	·3074	·3057	·3040	·3024	·3007	·2990	·2974	·2957	·2940	3	6	8	11	14
73°	·2924	·2907	·2890	·2874	·2857	·2840	·2823	·2807	·2790	·2773	3	6	8	11	14
74°	·2756	·2740	·2723	·2706	·2689	·2672	·2656	·2639	·2622	·2605	3	6	8	11	14
75°	·2588	·2571	·2554	·2538	·2521	·2504	·2487	·2470	·2453	·2436	3	6	8	11	14
76°	·2419	·2402	·2385	·2368	·2351	·2334	·2317	·2300	·2284	·2267	3	6	8	11	14
77°	·2250	·2233	·2215	·2198	·2181	·2164	·2147	·2130	·2113	·2096	3	6	9	11	14
78°	·2079	·2062	·2045	·2028	·2011	·1994	·1977	·1959	·1942	·1925	3	6	9	11	14
79°	·1908	·1891	·1874	·1857	·1840	·1822	·1805	·1788	·1771	·1754	3	6	9	11	14
80°	·1736	·1719	·1702	·1685	·1668	·1650	·1633	·1616	·1599	·1582	3	6	9	11	14
81°	·1564	·1547	·1530	·1513	·1495	·1478	·1461	·1444	·1426	·1409	3	6	9	12	14
82°	·1392	·1374	·1357	·1340	·1323	·1305	·1288	·1271	·1253	·1236	3	6	9	12	14
83°	·1219	·1201	·1184	·1167	·1149	·1132	·1115	·1097	·1080	·1063	3	6	9	12	14
84°	·1045	·1028	·1011	·0993	·0976	·0958	·0941	·0924	·0906	·0889	3	6	9	12	14
85°	·0872	·0854	·0837	·0819	·0802	·0785	·0767	·0750	·0732	·0715	3	6	9	12	14
86°	·0698	·0680	·0663	·0645	·0628	·0610	·0593	·0576	·0558	·0541	3	6	9	12	15
87°	·0523	·0506	·0488	·0471	·0454	·0436	·0419	·0401	·0384	·0366	3	6	9	12	15
88°	·0349	·0332	·0314	·0297	·0279	·0262	·0244	·0227	·0209	·0192	3	6	9	12	15
89°	·0175	·0157	·0140	·0122	·0105	·0087	·0070	·0052	·0035	·0017	3	6	9	12	15

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	0·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	0·0349	·0367	·0384	·0402	·0419	·0437	·0454	·0472	·0489	·0507	3	6	9	12	15
3°	0·0524	·0542	·0559	·0577	·0594	·0612	·0629	·0647	·0664	·0682	3	6	9	12	15
4°	0·0699	·0717	·0734	·0752	·0769	·0787	·0805	·0822	·0840	·0857	3	6	9	12	15
5°	0·0875	·0892	·0910	·0928	·0945	·0963	·0981	·0998	·1016	·1033	3	6	9	12	15
6°	0·1051	·1069	·1086	·1104	·1122	·1139	·1157	·1175	·1192	·1210	3	6	9	12	15
7°	0·1228	·1246	·1263	·1281	·1299	·1317	·1334	·1352	·1370	·1388	3	6	9	12	15
8°	0·1405	·1423	·1441	·1459	·1477	·1495	·1512	·1530	·1548	·1566	3	6	9	12	15
9°	0·1584	·1602	·1620	·1638	·1655	·1673	·1691	·1709	·1727	·1745	3	6	9	12	15
10°	0·1763	·1781	·1799	·1817	·1835	·1853	·1871	·1890	·1908	·1926	3	6	9	12	15
11°	0·1944	·1962	·1980	·1998	·2016	·2035	·2053	·2071	·2089	·2107	3	6	9	12	15
12°	0·2126	·2144	·2162	·2180	·2199	·2217	·2235	·2254	·2272	·2290	3	6	9	12	15
13°	0·2309	·2327	·2345	·2364	·2382	·2401	·2419	·2438	·2456	·2475	3	6	9	12	15
14°	0·2493	·2512	·2530	·2549	·2568	·2586	·2605	·2623	·2642	·2661	3	6	9	12	16
15°	0·2679	·2698	·2717	·2736	·2754	·2773	·2792	·2811	·2830	·2849	3	6	9	13	16
16°	0·2867	·2886	·2905	·2924	·2943	·2962	·2981	·3000	·3019	·3038	3	6	9	13	16
17°	0·3057	·3076	·3096	·3115	·3134	·3153	·3172	·3191	·3211	·3230	3	6	10	13	16
18°	0·3249	·3269	·3288	·3307	·3327	·3346	·3365	·3385	·3404	·3424	3	6	10	13	16
19°	0·3443	·3463	·3482	·3502	·3522	·3541	·3561	·3581	·3600	·3620	3	7	10	13	16
20°	0·3640	·3659	·3679	·3699	·3719	·3739	·3759	·3779	·3799	·3819	3	7	10	13	17
21°	0·3839	·3859	·3879	·3899	·3919	·3939	·3959	·3979	·4000	·4020	3	7	10	13	17
22°	0·4040	·4061	·4081	·4101	·4122	·4142	·4163	·4183	·4204	·4224	3	7	10	14	17
23°	0·4245	·4265	·4286	·4307	·4327	·4348	·4369	·4390	·4411	·4431	3	7	10	14	17
24°	0·4452	·4473	·4494	·4515	·4536	·4557	·4578	·4599	·4621	·4642	4	7	11	14	18
25°	0·4663	·4684	·4706	·4727	·4748	·4770	·4791	·4813	·4834	·4856	4	7	11	14	18
26°	0·4877	·4899	·4921	·4942	·4964	·4986	·5008	·5029	·5051	·5073	4	7	11	15	18
27°	0·5095	·5117	·5139	·5161	·5184	·5206	·5228	·5250	·5272	·5295	4	7	11	15	18
28°	0·5317	·5340	·5362	·5384	·5407	·5430	·5452	·5475	·5498	·5520	4	8	11	15	19
29°	0·5543	·5566	·5589	·5612	·5635	·5658	·5681	·5704	·5727	·5750	4	8	12	15	19
30°	0·5774	·5797	·5820	·5844	·5867	·5890	·5914	·5938	·5961	·5985	4	8	12	16	20
31°	0·6009	·6032	·6056	·6080	·6104	·6128	·6152	·6176	·6200	·6224	4	8	12	16	20
32°	0·6249	·6273	·6297	·6322	·6346	·6371	·6395	·6420	·6445	·6469	4	8	12	16	20
33°	0·6494	·6519	·6544	·6569	·6594	·6619	·6644	·6669	·6694	·6720	4	8	13	17	21
34°	0·6745	·6771	·6796	·6822	·6847	·6873	·6899	·6924	·6950	·6976	4	9	13	17	21
35°	0·7002	·7028	·7054	·7080	·7107	·7133	·7159	·7186	·7212	·7239	4	9	13	18	22
36°	0·7265	·7292	·7319	·7346	·7373	·7400	·7427	·7454	·7481	·7508	5	9	14	18	23
37°	0·7536	·7563	·7590	·7618	·7646	·7673	·7701	·7729	·7757	·7785	5	9	14	18	23
38°	0·7813	·7841	·7869	·7898	·7926	·7954	·7983	·8012	·8040	·8069	5	9	14	19	24
39°	0·8098	·8127	·8156	·8185	·8214	·8243	·8273	·8302	·8332	·8361	5	10	15	20	24
40°	0·8391	·8421	·8451	·8481	·8511	·8541	·8571	·8601	·8632	·8662	5	10	15	20	25
41°	0·8693	·8724	·8754	·8785	·8816	·8847	·8878	·8910	·8941	·8972	5	10	16	21	26
42°	0·9004	·9036	·9067	·9099	·9131	·9163	·9195	·9228	·9260	·9293	5	11	16	21	27
43°	0·9325	·9358	·9391	·9424	·9457	·9490	·9523	·9556	·9590	·9623	6	11	17	22	28
44°	0·9657	·9691	·9725	·9759	·9793	·9827	·9861	·9896	·9930	·9965	6	11	17	23	29

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	1·0000	1·0035	1·0070	1·0105	1·0141	1·0176	1·0212	1·0247	1·0283	1·0319	6	12	18	24	30
46°	1·0355	1·0392	1·0428	1·0464	1·0501	1·0538	1·0575	1·0612	1·0649	1·0686	6	12	18	25	31
47°	1·0724	1·0761	1·0799	1·0837	1·0875	1·0913	1·0951	1·0990	1·1028	1·1067	6	13	19	25	32
48°	1·1106	1·1145	1·1184	1·1224	1·1263	1·1303	1·1343	1·1383	1·1423	1·1463	7	13	20	26	33
49°	1·1504	1·1544	1·1585	1·1626	1·1667	1·1708	1·1750	1·1792	1·1833	1·1875	7	14	21	28	34
50°	1·1918	1·1960	1·2002	1·2045	1·2088	1·2131	1·2174	1·2218	1·2261	1·2305	7	14	22	29	36
51°	1·2349	1·2393	1·2437	1·2482	1·2527	1·2572	1·2617	1·2662	1·2708	1·2753	8	15	23	30	38
52°	1·2799	1·2846	1·2892	1·2938	1·2985	1·3032	1·3079	1·3127	1·3175	1·3222	8	16	24	31	39
53°	1·3270	1·3319	1·3367	1·3416	1·3465	1·3514	1·3564	1·3613	1·3663	1·3713	8	16	25	33	41
54°	1·3764	1·3814	1·3865	1·3916	1·3968	1·4019	1·4071	1·4124	1·4176	1·4229	9	17	26	34	43
55°	1·4281	1·4335	1·4388	1·4442	1·4496	1·4550	1·4605	1·4659	1·4715	1·4770	9	18	27	36	45
56°	1·4826	1·4882	1·4938	1·4994	1·5051	1·5108	1·5166	1·5224	1·5282	1·5340	10	19	29	38	48
57°	1·5399	1·5458	1·5517	1·5577	1·5637	1·5697	1·5757	1·5818	1·5880	1·5941	10	20	30	40	50
58°	1·6003	1·6066	1·6128	1·6191	1·6255	1·6319	1·6383	1·6447	1·6512	1·6577	11	21	32	43	53
59°	1·6643	1·6709	1·6775	1·6842	1·6909	1·6977	1·7045	1·7113	1·7182	1·7251	11	23	34	45	56
60°	1·7321	1·7391	1·7461	1·7532	1·7603	1·7675	1·7747	1·7820	1·7893	1·7966	12	24	36	48	60
61°	1·8040	1·8115	1·8190	1·8265	1·8341	1·8418	1·8495	1·8572	1·8650	1·8728	13	26	38	51	64
62°	1·8807	1·8887	1·8967	1·9047	1·9128	1·9210	1·9292	1·9375	1·9458	1·9542	14	27	41	55	68
63°	1·9626	1·9711	1·9797	1·9883	1·9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64°	2·0503	2·0594	2·0686	2·0778	2·0872	2·0965	2·1060	2·1155	2·1251	2·1348	16	31	47	63	78
65°	2·1445	2·1543	2·1642	2·1742	2·1842	2·1943	2·2045	2·2148	2·2251	2·2355	17	34	51	68	85
66°	2·2460	2·2566	2·2673	2·2781	2·2889	2·2998	2·3109	2·3220	2·3332	2·3445	18	37	55	73	92
67°	2·3559	2·3673	2·3789	2·3906	2·4023	2·4142	2·4262	2·4383	2·4504	2·4627	20	40	60	79	99
68°	2·4751	2·4876	2·5002	2·5129	2·5257	2·5386	2·5517	2·5649	2·5782	2·5916	22	43	65	87	108
69°	2·6051	2·6187	2·6325	2·6464	2·6605	2·6746	2·6889	2·7034	2·7179	2·7326	24	47	71	95	119
70°	2·7475	2·7625	2·7776	2·7929	2·8083	2·8239	2·8397	2·8556	2·8716	2·8878	26	52	78	104	130
71°	2·9042	2·9208	2·9375	2·9544	2·9714	2·9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	144
72°	3·0777	3·0961	3·1146	3·1334	3·1524	3·1716	3·1910	3·2106	3·2305	3·2506	32	64	96	129	161
73°	3·2709	3·2914	3·3122	3·3332	3·3544	3·3759	3·3977	3·4197	3·4420	3·4646	36	72	108	144	180
74°	3·4874	3·5105	3·5339	3·5576	3·5816	3·6059	3·6305	3·6554	3·6806	3·7062	41	81	122	163	204
75°	3·7321	3·7583	3·7848	3·8118	3·8391	3·8667	3·8947	3·9232	3·9520	3·9812					
76°	4·0108	4·0408	4·0713	4·1022	4·1335	4·1653	4·1976	4·2303	4·2635	4·2972					
77°	4·3315	4·3662	4·4015	4·4374	4·4737	4·5107	4·5483	4·5864	4·6252	4·6646					
78°	4·7046	4·7453	4·7867	4·8288	4·8716	4·9152	4·9594	5·0045	5·0504	5·0970					
79°	5·1446	5·1929	5·2422	5·2924	5·3435	5·3955	5·4486	5·5026	5·5578	5·6140					
80°	5·6713	5·7297	5·7894	5·8502	5·9124	5·9758	6·0405	6·1066	6·1742	6·2432	Mean differences not sufficiently accurate.				
81°	6·3138	6·3859	6·4596	6·5350	6·6122	6·6912	6·7720	6·8548	6·9395	7·0264					
82°	7·1154	7·2066	7·3002	7·3962	7·4947	7·5958	7·6996	7·8062	7·9158	8·0285					
83°	8·1443	8·2636	8·3863	8·5126	8·6427	8·7769	8·9152	9·0579	9·2052	9·3572					
84°	9·5144	9·6768	9·8448	10·019	10·199	10·385	10·579	10·780	10·988	11·205					
85°	11·430	11·664	11·909	12·163	12·429	12·706	12·996	13·300	13·617	13·951					
86°	14·301	14·669	15·056	15·464	15·895	16·350	16·832	17·343	17·886	18·464					
87°	19·081	19·740	20·446	21·205	22·022	22·904	23·859	24·898	26·031	27·271					
88°	28·636	30·145	31·821	33·694	35·801	38·188	40·917	44·066	47·740	52·081					
89°	57·290	63·657	71·615	81·847	95·489	114·59	143·24	190·98	286·48	572·96					

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	− ∞	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	$\bar{3}$·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196	Differences not sufficiently accurate.				
1°	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2°	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3°	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4°	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315					
5°	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{1}$·0046	$\bar{1}$·0120	13	26	39	52	65
6°	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7°	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8°	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9°	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10°	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11°	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12°	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13°	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14°	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15°	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16°	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17°	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18°	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19°	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20°	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21°	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22°	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23°	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24°	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25°	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26°	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27°	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28°	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29°	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30°	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31°	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32°	$\bar{1}$·7242	7254	7266	7273	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33°	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34°	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35°	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36°	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37°	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38°	$\bar{1}$·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39°	$\bar{1}$·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40°	$\bar{1}$·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41°	$\bar{1}$·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42°	$\bar{1}$·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43°	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44°	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46°	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47°	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48°	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49°	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50°	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51°	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52°	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53°	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54°	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55°	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56°	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57°	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58°	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59°	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60°	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61°	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62°	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63°	$\bar{1}$·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64°	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65°	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66°	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67°	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68°	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69°	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70°	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71°	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72°	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73°	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74°	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75°	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76°	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77°	$\bar{1}$·9887	9889	9891	9893	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78°	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79°	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80°	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81°	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82°	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83°	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84°	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85°	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86°	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87°	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88°	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89°	$\bar{1}$·9999	9999	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0	0	0	0	0

LOGARITHMS OF COSINES.

Subtract Differences.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	9999	0	0	0	0	0
1°	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2°	$\bar{1}$·9997	9997	9997	9997	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3°	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4°	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5°	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6°	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7°	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8°	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9°	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10°	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11°	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12°	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13°	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14°	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15°	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16°	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17°	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18°	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19°	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20°	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21°	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22°	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23°	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24°	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25°	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26°	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27°	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28°	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29°	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30°	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31°	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32°	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33°	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34°	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35°	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36°	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37°	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38°	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39°	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40°	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41°	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42°	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43°	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44°	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

Subtract Differences

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46°	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47°	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48°	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49°	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50°	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51°	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52°	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53°	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54°	$\bar{1}$·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55°	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56°	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57°	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58°	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59°	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60°	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61°	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62°	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63°	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64°	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65°	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66°	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67°	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68°	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69°	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70°	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71°	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72°	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73°	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74°	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75°	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76°	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77°	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78°	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79°	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80°	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81°	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82°	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83°	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84°	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	$\bar{2}$·9489	13	26	39	52	66
85°	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86°	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87°	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88°	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89°	$\bar{2}$·2419	$\bar{2}$·196	$\bar{2}$·145	$\bar{2}$·087	$\bar{2}$·020	$\bar{3}$·941	$\bar{3}$·844	$\bar{3}$·719	$\bar{3}$·543	$\bar{3}$·242					

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	$-\infty$	$\bar{3}.242$	$\bar{3}.543$	$\bar{3}.719$	$\bar{3}.844$	$\bar{3}.941$	$\bar{2}.020$	$\bar{2}.087$	$\bar{2}.145$	$\bar{2}.196$					
1°	$\bar{2}.2419$	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2°	$\bar{2}.5431$	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3°	$\bar{2}.7194$	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4°	$\bar{2}.8446$	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5°	$\bar{2}.9420$	9506	9591	9674	9756	9836	9915	9992	$\bar{1}$ 0068	$\bar{1}.0143$	13	26	40	53	66
6°	$\bar{1}.0216$	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7°	$\bar{1}.0891$	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8°	$\bar{1}.1478$	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9°	$\bar{1}.1997$	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10°	$\bar{1}.2463$	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11°	$\bar{1}.2887$	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12°	$\bar{1}.3275$	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13°	$\bar{1}.3634$	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14°	$\bar{1}.3968$	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15°	$\bar{1}.4281$	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16°	$\bar{1}.4575$	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17°	$\bar{1}.4853$	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18°	$\bar{1}.5118$	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19°	$\bar{1}.5370$	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20°	$\bar{1}.5611$	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21°	$\bar{1}.5842$	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22°	$\bar{1}.6064$	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23°	$\bar{1}.6279$	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24°	$\bar{1}.6486$	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25°	$\bar{1}.6687$	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26°	$\bar{1}.6882$	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27°	$\bar{1}.7072$	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28°	$\bar{1}.7257$	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29°	$\bar{1}.7438$	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30°	$\bar{1}.7614$	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31°	$\bar{1}.7788$	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32°	$\bar{1}.7958$	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33°	$\bar{1}.8125$	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34°	$\bar{1}.8290$	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35°	$\bar{1}.8452$	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36°	$\bar{1}.8613$	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37°	$\bar{1}.8771$	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38°	$\bar{1}.8928$	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39°	$\bar{1}.9084$	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40°	$\bar{1}.9238$	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41°	$\bar{1}.9392$	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42°	$\bar{1}.9544$	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43°	$\bar{1}.9697$	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44°	$\bar{1}.9848$	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	0.0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46°	0.0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47°	0.0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48°	0.0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49°	0.0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50°	0.0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51°	0.0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52°	0.1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53°	0.1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54°	0.1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55°	0.1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56°	0.1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57°	0.1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58°	0.2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59°	0.2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60°	0.2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61°	0.2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62°	0.2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63°	0.2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64°	0.3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	7	10	13	16
65°	0.3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66°	0.3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67°	0.3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68°	0.3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69°	0.4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70°	0.4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71°	0.4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72°	0.4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73°	0.5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74°	0.5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75°	0.5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76°	0.6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77°	0.6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78°	0.6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79°	0.7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80°	0.7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81°	0.8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82°	0.8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83°	0.9109	9173	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84°	0.9784	9857	9932	1.0008	1.0085	1.0164	1.0244	1.0326	1.0409	1.0494	13	26	40	53	66
85°	1.0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	65	81
86°	1.1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87°	1.2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88°	1.4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89°	1.7581	1.804	1.855	1.913	1.980	2.059	2.156	2.281	2.457	2.758					

ANSWERS

EXERCISE I

Section A. Percentages

p. 13.

1. (1) **1·6.** (2) **208·8.** (3) **3·6936.** (4) 12*s.* 9*d.*
2. (1) **31·25.** (2) **37·5.** (3) **46·67.** (4) **18·33.**
3. (1) **42·5.** (2) 7·42. (3) **36·03.** (4) **50.**
4. (1) **356·25.** (2) 150 gm.
5. 25 ft. 6. **4·725** cwt.
7. **37·7** lb guncotton; **17·4** lb nitroglycerine; **2·9** lb min jelly.
8. **7·83.** 9. **36·06.** 10. **47·87.** 11. **620.** 12. $37\frac{1}{2}$ lb.
13. Men, **54·02%**; women, **39·31%**; children, **6·67%**.
14. Copper, **68·9%**; zinc, **19·6%**; lead, **6·3%**.

Section B. Ratio

p. 14.

1. (1) $\frac{7}{40}$. (2) $\frac{500}{197}$. (3) $\frac{22}{15}$. (4) $\frac{5}{11}$.
2. (1) 1·2. (2) 0·53. (3) 1·18. (4) 0·61224.
3. 20·6.
4. 15·1.
5. $\frac{5}{16}$ greatest; $\frac{7}{32}$ least.
6. 3·9 in.; 5·2 in.
7. **3·43 : 1.**
8. **4·32** in.
9. **6·25 : 6·6.**

Section C. Proportion

p. 15.

1. (1) $85\frac{5}{7}$. (2) 135. (3) 38·9. (4) 68·75.
2. £22 10*s.*
3. 21*s.* $0\frac{1}{2}$*d.*
4. £7 19*s.* 6*d.*
5. (1) 20. (2) 21.
6. 4·5.
7. $7\frac{4}{5}$ gal.
8. £1 7*s.* 1*d.*
9. $15\frac{3}{5}$ in.

Section D. Approximations

p. 20.

1. (1) (*a*) 18·72; (*b*) 18·7160. (2) (*a*) 0·007204; (*b*) 0·00720.
2. (1) £3,870,000,000. (2) 3,866,100,000.
 (3) £3,866,122,000.
3. (*a*) 39·998. (*b*) 40·00 (*c*) 40.
4. (*a*) 0·0005. (*b*) 0·05 lb. (*c*) 0·0005 ft.
 (*d*) 0·005 in. (*e*) 0·05 mile.
5. (1) 0·64. (2) 0·9. (3) 4·5. (4) 7·2. (5) 0·6. (6) 0·006.
6. 39·2 and 39·6. 7. $\pm$ 0·000475.
8. 6·9 cu ft to 2 sig figs. 7·0 cu ft *correct* to 2 sig figs.
9. $\pi = \frac{22}{7}$ is correct to 3 sig figs and approximately correct to 4 sig figs.
10. (1) 17·0. (2) 36·4. (3) 2·45. (4) 11·5.
11. 0·56.

Section E. Square Root

p. 22.

1. (1) 110. (2) 440.
2. (1) 0·6. (2) 0·16. (3) 1·65. (4) 0·36.
3. (1) 572. (2) 135.
4. (1) 56·89. (2) 26·74 (3) 26·35. (4) 2·278.
5. (1) 0·9555. (3) 0·2369. (5) 0·3027.
 (2) 0·7187. (4) 0·632.
6. 1·4142; 1·212. 7. (1) 6·5. (2) 18. 8. (1) 11·26. (2) 8·025.
9. 9·46. 10. 22·85. 11. (1) $\frac{7}{11}$. (2) $\frac{19}{25}$. (3) $\frac{80}{17}$. 12. 3·65.

13. (1) 0·707. (2) 1·155. (3) 1·443.
14. (1) 1·225. (2) 0·632. (3) 0·3162.
15. 2·121. 16. 1·284. 17. 0·236. 18. 5·656.

Miscellaneous Exercises

p. 24.

1. (*a*) 1·56% correct to 3 sig figs. (*b*) 220·6.
2. Copper 69·03%, zinc 19·63%, lead 6·28%.
3. (i) By arithmetic 93·76 acres; to three sig figs. 93·8 acres.
 (ii) 19·83 lb per sq in.
5. Time 6 hr. Average speed 10·8 m.p.h. approx.
6. 192 sq ft. 7. $17\frac{1}{7}$ cu ft.
8. (*a*) 43·50 litres is 2655 cu in.
 (*b*) £308 19*s*. 5*d*.; £646 0*s*. 7*d*.

9. (*a*) 702 to sig 3 figs. (*b*) 1·54 to 3 sig figs.
 (*c*) 1·34 to 3 sig figs.
10. By arithmetic 177·32; to 4 sig figs as in data, 177·3 in.
11. (*a*) 3·294 miles.
 (*b*) 3 miles 4 furlongs to nearest furlong.
 (*c*) 0·79% or 0·8% very nearly under size.
12. $14\frac{7}{12}$ in. 648 r.p.m., 499 r.p.m., 405 r.p.m. to three sig figs. Belt slip makes third unreliable.
13. 88·8 lb copper; 22·8 lb lead; 8·4 lb tin.
14. Number 3 is 1·2 cm above the average of 116·3, and must be rejected.
15. Average speed for journey is 32 m.p.h. Actual speed for last stage is 48 m.p.h.
16. 1 in 25 or 4% oversize for any length. 17. 5 ft.
18. (*a*) $627\frac{1}{2}$. (*b*) £24 down payment.
 (*c*) £8 3*s*. 3*d*. per month to nearest penny over.
19. (*a*) Average speed for 9 laps, 79·1 m.p.h. to 3 sig figs.
 (*b*) $107\frac{2}{3}$ lb total weight.
20. (*a*) 160 ml; 12·5 gal. (*b*) 6 mins.
 (*c*) 11,482 ft 11 in. by arithmetic, say 11,500 ft to 3 sig figs.
21. He must make 430 articles per week. Wages cost 20*s*. 0*d*. per 100.

EXERCISE 11

p. 42.

1. (1) $43\frac{1}{8}$ sq in.
 (2) 251·68 sq in.
 (3) 1·83 acres.
 (4) 33·02 sq in.
 (5) 3 acres, 1320 sq yd.
2. 487 sq yd 1 sq ft.
3. 592 sq in. 4. 90.
5. $83\frac{1}{3}$ ft.
6. £3 2*s*. 7*d*.
7. $5\frac{20}{21}$ cm.
8. (*a*) 18·88 sq in.
 (*b*) 4464 sq yd.
 (*c*) 48·45 sq cm.
9. 6·93 sq in.
10. 19·2 in.
11. 5·43 sq in.
12. 908 sq ft.
13. $B = \sqrt{D^2 + 4DH}$.
14. (*a*) $4x$ in.
 (*b*) $2(m + n)$ in.
 (*c*) $(6a + 4b)$ in.
15. $(6750 - ab)$ sq ft.
16. $\dfrac{144\,mn}{lb}$
17. 0·703 p gm per sq mm.
18. (ii) 5·719 gal by arithmetic, 5·72 gal to 3 sig figs.
 (iii) 5·1136 lb by arithmetic, 5·1 lb to 2 sig figs.

19. $3\frac{\sqrt{3r^2}}{2}$. 20. 39·375 sq in.

21. 2·771 in., 13·86 sq in. 22. 19·6% saving (to 3 sig figs).
23. (i) 187 double strokes.
    (ii) Ratio $\frac{\text{m'cd area}}{\text{swept area}} = 87\%$ (to 3 sig figs).
24. 5·175 min by arithmetic, say, 5·2 min to 2 sig figs. 0·0153 in. or 15 thousandths per revolution.
25. 18·36 sq in.
26. (i) 4·25 sq in. (ii) 0·054 sq in. or 1·27%. (iii) $t(a + b) - t^2$.

Miscellaneous Exercises

p. 47.

1. 47·1 cu ft.
2. (*a*) 1716 by arithmetic, 1700 cu in. to 2 sig figs. (*b*) 25%.
3. (*a*) 5·105°. (*b*) diag 13 cm and 7·1 cm approx; vol = 0·0003 cu m.
4. (*a*) alt = 1 ft 7·2 in.
   (*b*) 9·116 by arithmetic, D = 9·12 in. to 3 sig figs.
5. Numerical answer inadequate.
6. $V = t\left\{\frac{\pi r^2}{2} + h(d + 2r)\right\}$. 7. $\frac{l + x}{v}$ sec.
8. 29·7 *lb* to 3 sig figs.
9. $D = \sqrt{d^2 + \frac{4V}{\pi l}}$; 1414 cu ft.
10. Numerical answer inadequate.
11. $\frac{3a}{4}$; 1 : 0·56 to 2 sig figs.
12. (*b*) 20·90 by arithmetic, 21 tons to 2 sig figs.
13. 5·878 by arithmetic, 5·9 cm to 2 sig figs.
14. 656,000 ft lb to 3 sig figs. 15. 312 sq yds.
16. (*a*) 7220 gal to 3 sig figs. (*b*) 116,000 gal.
17. (*a*) 652 ft to 3 sig figs. (*b*) 7·66 lb to 3 sig figs.
18. 1760 gal/min to 3 sig figs.

EXERCISE III

Section A

p. 67.

1. (*a*) $3a$, $\frac{1}{2}y$, $1 + 5b$. (*b*) $5c$, $3x^2$, $\frac{5}{1 + x}$. (*c*) $y + z$.
2. $8x + 8{\cdot}5y + 15z$; 70.

3. (*a*) $6x^3y^3$. (*b*) $\frac{3}{4}pq^3$. (*c*) $24a^2b^2c^2$. (*d*) $\frac{1}{15}a^3bt^3$; 400.
4. (*a*) $\frac{35a + 69b + 86c}{30}$. (*b*) $\frac{47}{6a}$.
5. (*a*) $7xy$. (*b*) $\frac{7a}{4}$. (*c*) $\frac{6x^2y}{5}$. (*d*) $\frac{21b^2}{10ac}$.
6. (1) $\frac{3}{2a}$. (2) $\frac{3c^2}{4}$. (3) $\frac{5ax^2}{4}$. (4) $\frac{2}{3x^2}$.
7. (*a*) $196x^2$. (*b*) $36a^4$. (*c*) $\frac{4}{9}x^4y^2$. (*d*) $\frac{25x^2}{9y^4}$.
8. (*a*) $\pm 3xy$. (*b*) $\pm 8ay^3$. (*c*) $\pm\frac{2}{3}x^2y^2$. (*d*) $\pm\frac{4xyz}{3}$.
9. $\alpha = 0{\cdot}00001$. 10. Numerical answer inadequate.
11. d^3. 12. (1) c^2. (2) ab.

Section B

p. 69.

13. (1) $2a$. (7) -1. (13) 24
    (2) 0. (8) -1. (14) $\frac{3a}{2}$
    (3) $-2a$ (9) 1.
    (4) $-a^2$ (10) -5. (15) $-3x^2$.
    (5) $-a^2$. (11) 5. (16) $-10x^3$.
    (6) a^2. (12) -5.
14. (*a*) $\frac{32n^2 - 25n + 63}{30}$. (*c*) $\frac{-2a + 31b}{6}$.
    (*b*) $\frac{19x - 3y}{7}$. (*d*) $\frac{7x - 5y - 6z}{3}$.
15. 12·75. 16. 20·125
17. (*a*) $1 - x - x^2 = 1 - x(1 + x)$.
    (*b*) $5x^2 - 7x + 14 = 5x^2 - 7(x - 2)$.
    (*c*) $3a - 2b + 4c + 7a = 10a - 4(\frac{1}{2}b - c)$.
18. (i) $\frac{x}{y}$; (ii) $\frac{4}{x - 1}$.
19. (i) x^2; (ii) $4(x - 3)$; (iii) $4a^2 + 12ab + 9b^2$; (iv) $\frac{7 - 3a}{(1 - a)^2}$;
    (v) $8y^2 - 10y - 12$.
20. (*a*) $\frac{D}{1760V}$ hr; 4·5 hr. (*b*) R = 1·7 to 2 sig figs.

EXERCISE IV

Section A

p. 83.

1. $ap + aq - ay$.
2. $3mnab - 3mncd + 3mnda$.
3. $15x^2m - 10x^2n + 25x^2p$.
4. $5R^3 - 5R^2 + 5R$.
5. $\frac{1}{2}x^3 - x^2 + \frac{1}{2}x$.
6. $\frac{4a^3}{3} - a^2 + \frac{7a}{3}$.
7. $2a^2 + 11ab + 12b^2$.
8. $3x^2 - xy - 2y^2$.
9. $x^2 - 5{\cdot}7x + 8$.
10. $2p^2 - 0{\cdot}7p - 4{\cdot}9$.
11. $4 - 0{\cdot}9y - y^2$.
12. $10p^2q^2 - 17pqmn + 3m^2n^2$.
13. $35 - 12a - 32a^2$.
14. $18p^4 + 3p^2q^2 - q^4$.
15. $2R^2 - R - 6$.
16. $195 + 34q - 8q^2$.
17. $35a^2b^2 - 26abx - 16x^2$.
18. $25p^2 + 30pq + 9q^2$.
19. $a^2 - 4ab + 4b^2$.
20. $16m^2 + 24mn + 9n^2$.
21. $225p^2 - 30pq + q^2$.
22. $1 - p^2 + 3q - pq + 2q^2$.
23. $9x^2 - 6xy + y^2 - 6mx + 2my - 3m^2$.
24. $a^2 + 2ab + b^2 + ac + bc - 2c^2$.
25. $R^2 - x^2$.
26. $4c^2 - d^2$.
27. $25m^2n^2 - 16p^2q^2$.
28. $\frac{1}{a^2} - x^2$.
29. $4c^2 - d^2$.
30. $\frac{9p^2q^2}{4} - \frac{16c^2}{25}$.
31. $x^2 - 2{\cdot}25$.
32. $6{\cdot}25a^2 - 1{\cdot}96$.
33. $p^2 + 4q^2 + c^2 - 2pc + 4pq - 4qc$.
34. $9a^2 + 4b^2 + 16c^2 - 12ab - 24ac + 16bc$.

Section B

p. 84.

1. $a - 2$.
2. $x + 1$.
3. $2x - p$.
4. $5a - 3b$.
5. $x - 3$.
6. $2mn + 3a$.
7. $a + p$.
8. $2a - 3b$.
9. $\frac{1}{x} - \frac{1}{a}$.
10. $\frac{2p}{3} + a$.

Section C

p. 85.

I

1. $x(a - b + c)$.
2. $pq(pq - ay + by)$.
3. $7a(2a^2 - ay + 8y^2)$.
4. $9abc(6a^2b - 4b^2c + 3c^2)$.
5. $\frac{c}{x^2}\left(a - \frac{b}{x} + \frac{d}{x^2}\right)$.
6. $\frac{2m}{n}\left(\frac{5p}{q} - \frac{q}{p} + \frac{2p}{r}\right)$.

II

1. $(a + b)(x + y)$.
2. $(ac - d)(ac + b)$.
3. $(x^2 + b^2)(c - d)$.
4. $(a^2 + 2)(2a - 3)$.
5. $(a + 5)(11a^2 + 7)$.
6. $(mn - pq)(a^2 + b^2)$.
7. $(ax - b)(bx - a)$.
8. $(m^2 + 1)(a - b)$.
9. $(x^2 + 1)(2x - 1)$.
10. $(p^2 - q)(1 + r)$.
11. $(a - 3c)(a - 2b)$.

III

1. $(a + 5)(a + 4)$.
2. $(a - 3)(a - 3)$.
3. $(m + 2n)(m - 3n)$.
4. $(x + 7)(x - 5)$.
5. $(x - 7)(x + 2)$.
6. $(x + 9)(x - 8)$.
7. $(3 + a)(7 + a)$.
8. $(1 - a)(1 - 2a)$.
9. $(p + 9)(p - 5)$.
10. $(x + m)(x + n)$.
11. $(x - 5y)(x - 17y)$.
12. $(3a - 1)(a - 2)$.
13. $(2a - 3)(2a - 5)$.
14. $(4a + 5)(5a + 4)$.
15. $(3a - 7)(3a + 4)$.
16. $(7p - 4)(2p - 3)$.
17. $(3a + b)(4a + 5b)$.
18. $(13r - 1)(2r - 3)$.

IV

1. $(2a - 3b)^2$.
2. $(5a - 6b)^2$.
3. $(7m + 2n)^2$.
4. $(p + 2)^2$.
5. $(q - 4)^2$.
6. $(x - \frac{1}{2})^2$.
7. $\left(\frac{1}{a} + \frac{1}{b}\right)^2$.
8. $(R - 1)^2$.
9. $(a + 3b)(a - 3b)$.
10. $(5x + 7y)(5x - 7y)$.
11. $(11x + 4y)(11x - 4y)$.
12. $\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{a} - \frac{1}{b}\right)$.
13. $(x + \frac{1}{2})(x - \frac{1}{2})$.
14. $(1 + \frac{3}{4}x)(1 - \frac{3}{4}x)$.
15. $(\frac{3}{4} + 2a)(\frac{3}{4} - 2a)$.
16. $(12p + 13q)(12p - 13q)$.
17. $(a + m + n)(a - m - n)$.
18. $-(p + q)(3p + q)$.
19. $(p - q + r)(p - q - r)$.
20. $b(2a + b)$.
21. $(a + b + c)(a + b - c)$.
22. $(m + n + p)(m - n - p)$.
23. $(x + a - b)(x - a + b)$.

V

1. $(m - 3n)(m^2 + 3mn + 9n^2)$.
2. $8(a - 2b)(a^2 + 2ab + 4b^2)$.
3. $\left(\frac{1}{m} + \frac{1}{n}\right)\left(\frac{1}{m^2} - \frac{1}{mn} + \frac{1}{n^2}\right)$.
4. $(R + 1)(R^2 - R + 1)$.
5. $2(a + 3)(a + 4)$.
6. $5a(3x - 1)(x - 2)$.
7. $4(1 + 2x)(5 - x)$.
8. $3m(4x - 5)(4x + 3)$.

Section D

p. 87.

1. $\frac{b}{a}$. 2. $\frac{a-7}{a-3}$. 3. $q(p+q)$. 4. $\frac{a-5}{a-6}$. 5. m.

Section E

p. 87.

1. $\frac{5a}{6x}$.
2. $\frac{m^2+n^2+q^2}{mnq}$.
3. $\frac{2x}{(x+1)(x-1)}$.
4. $\frac{-2ab}{(a+b)(a-b)}$.
5. $\frac{12x}{(x-3)(x+3)}$.
6. $\frac{7-3a}{(1-a)^2}$.
7. 0.
8. $\frac{2ab}{a^3-8b^3}$.
9. $\frac{4p}{(p+q)(p-q)}$.
10. $\frac{1}{(a-4)(a-1)(a-3)}$.

Section F. Miscellaneous

p. 87.

1. (1) $9(2x+3y)(2x-3y)$.
(2) $(4x-3y)(x+2y)$. (3) $ab(a-2)(a-1)$.
2. (1) $(7{\cdot}4\times 13^2)+(7{\cdot}4\times a^2)=7{\cdot}4(13^2+a^2)$.
(2) $\frac{2x-y}{x-y}=\frac{2(y-2x)}{2(y-x)}$.
(3) $$\begin{aligned}3a^2+5ab-2b^2&=(3a^2+6ab)-(ab+2b^2)\\&=3a(a+2b)-b(a+2b)\\&=(3a-b)(a+2b).\end{aligned}$$
3. Vol 760 cu in.; wt 200 lb, both to 2 sig figs.
4. (1) a^2+b^2. (2) $p^2+2pq+q^2$. (3) $\frac{3}{5+x^2}$.
5. (1) $2xy$. (2) $K-h$. 6. (1) $u+v$. (2) $r-t+s$.
7. $a^4+4a^2x^2+16x^4$. 8. (a) $\frac{x}{x^2-16}$; (b) $\frac{x+1}{x+2}$.
9. $(2x+y+3a-2b)(2x+y-3a+2b)$;
$(3a+b)(a+5b-2c)$.
10. (a) $(3x+2)(x-3)$. (b) $(4a+7b^2)(4a-7b^2)$.
(c) $(b+3)(a-2)$.

11. (i) 18. (ii) $4b(a+b)$; $(x-5)(x+2)$;
$(x+y)(x-y)(a+b)$.
12. (a) $\frac{2x+13}{4x^2-8x-5}$.
(b) (i) $(3a+4b)(3a-4b)$; (ii) $(2x-7)(x+3)$.
(c) $x(x-1)$; $a=\pm 1$.
13. (a) (i) $\left(2x+\frac{3}{4x}\right)\left(2x-\frac{3}{4x}\right)$; (ii) $(a-2)(b+c)$.
(b) (i) $\frac{a}{a-3b}$; (ii) $\frac{-4x}{(x+1)(x-1)}$.
14. (i) $\frac{pr}{q^2}$; (ii) $x^2-xy-6y^2$; (iii) $\frac{a-b}{ab}$.
15. $\frac{-2s}{(r+s)(r-s)}$.
16. (a) $\frac{5p-q}{(p+q)(p-q)}$. (b) $\frac{5(p+q)(p-q)}{5p-q}$.
17. $(2a-x)(a+3x)$.
18. (1) $\frac{\pi}{4}(D+d)(D-d)(D^2+d^2)$.
(2) $(3m+5)(2m+3)$.
19. 2. 20. $\frac{4abf}{(a^2-b^2)^2}$, $\frac{4bf}{a^3}$. 21. (1) $a+b$. (2) $\frac{a}{x}$.

EXERCISE V

A. Simple Equations

p. 109.

1. 12.
2. 3.
3. 4.
4. 17.
5. 14.
6. $\frac{1}{5}$.
7. (a) 2; (b) 2; (c) 2.
8. -5.
9. 0·8.
10. 2·7.
11. -15.
12. 2·9.
13. 1·18.
14. 3.
15. $2\frac{7}{19}$.
16. V = 36, R = 8.
17. $16\frac{1}{2}$.
18. $\frac{2}{3}$.
19. 1·8.
20. $9\frac{1}{4}$.
21. 2·5.
22. $-\frac{3}{5}$.
23. 620.
24. $8\frac{8}{19}$.
25. $-4\frac{4}{11}$.
26. $p=-\frac{43}{44}$.
27. 62°, 65°, 53°.
28. 11·9 in., 10·1 in.
29. 2.
30. $\frac{13}{24}$.
31. $9\frac{1}{3}$ in.
32. 18·84.
33. 528·9.
34. $\frac{3}{5}$.
35. 7.
36. $\frac{1}{2}$.
37. 11 min, 4 min.
38. 8.
39. 39·1.

B. Simultaneous Equations

p. 111.

1. $x = 1, y = 1.$
2. $x = 3, y = 1.$
3. $x = -4, y = 2\frac{1}{2}.$
4. $x = 3, y = -5.$
5. $x = 9, y = 12.$
6. $x = 12, y = 2.$
7. $x = 6, y = -4.$

C. Miscellaneous Problems and Equations

p. 112.

1. $P = 1{\cdot}8, Q = 0{\cdot}32.$
2. $P = 3, Q = 4.$
3. $\frac{1}{x} = 77, \frac{1}{y} = 48.$
4. 55 and 95.
5. 154*s*. 77*s*.
6. (*a*) $x = 5\frac{1}{3}$; (*b*) $x(x - y)$; (*c*) base 10, sides each 15.
7. (i) $\frac{-3 \pm \sqrt{7}}{2}.$ (ii) $x = 4, y = 3.$
8. (*a*) $x = 3, y = \frac{1}{3}$; (*b*) $A = 2\frac{6}{7}, B = 175.$
9. Speeds are 38 and 22 m.p.h.
10. Holes could be 4·12 in. dia.
11. $x = 3, y = \frac{1}{3}.$
12. $a = 0{\cdot}5, b = 0{\cdot}6, E = 6{\cdot}5.$
13. $a = -1{\cdot}36, b = 1{\cdot}38.$
14. $x = 40, y = 16.$
15. $P = 4, \frac{Q}{P} = 2, 3.$
16. $\frac{1}{x} = 4, \frac{1}{y} = 3, \frac{1}{7}.$
17. $x = 3, y = \frac{1}{2}.$

EXERCISE VI

I. Construction of Formulæ

p. 121.

1. (*a*) $\frac{16c}{ab}.$
   (*b*) $\frac{144c}{ab}.$
2. (*a*) $x + 4.$ (*b*) $x^2 + 8x + 16.$
3. $0{\cdot}0225K$ m.p.h.
4. $11{,}200x.$
5. (1) $\frac{76}{45}x$ ft per sec.
   (2) $70{\cdot}3p.$

II. Evaluation of Formulæ

p. 122.

1. 52427.
2. 0·284.
3. 338·6.
4. 81·5° F.
5. 4·02.
6. 0·616.
7. 0·47.
8. $f = 10.$
9. 85·6.

III. Changing the Subject of a Formula

p. 123.

1. $E = CR + e$, 245.
2. (*a*) $f = \frac{16T}{\pi d^3}.$ (*b*) $d = \sqrt[3]{\frac{16T}{\pi f}}.$
3. $C = \frac{825H}{E}.$
4. (1) $d = \sqrt{\frac{H}{0{\cdot}5(r + 1)}}.$ (2) $r = \frac{2H - d^2}{d^2}.$
5. $n = \frac{cq - aq - b}{p(c - a)}.$
6. $V = \frac{RE}{R + r}$, V is doubled.
7. $V = \sqrt{\frac{2hDg}{0{\cdot}03L}}$, 1·4.
8. 1·18.
9. 1·78.
10. $n = \sqrt{\frac{NR - 1}{r}}$, 7·17.
11. $f = \frac{p(D^2 + d^2)}{D^2 - d^2}$, 2467 per sq in.
12. $B = \sqrt{\frac{112 \times 10^5 F}{A}}$, 8·7 sq cm.
13. $f = \frac{v^2 - u^2}{2s}.$
14. $P = \left(\frac{2b + a}{2a + b}\right)Q$, 216·4.
15. (*a*) 44·76. (*b*) $n = 6.$
16. $u = \frac{2s - ft^2}{2t}$, $-8.$
17. $l = \sqrt[3]{\frac{48EId}{W}}.$
18. $D = \sqrt[4]{\frac{583TL}{NA}}.$

Miscellaneous

p. 126.

1. Merit best judged by method of approach.
2. $l = 0{\cdot}1886ab^2.$
3. $4{\cdot}6d^2l$ in. to 2 sig figs.
4. (*a*) $x = \pm\sqrt{\frac{y^2gL}{4\pi^2} - L^2}$; (*b*) $x = \sqrt[3]{a^3 - \frac{3y}{4\pi}}.$
   (*c*) $x = (y - 1)^{2/5}.$
5. $L = 81{\cdot}5.$
6. $h = \frac{4}{3}R.$
7. $h = \pm\sqrt{\frac{s^2}{\pi^2 r^2} - r^2}.$
8. (ii) $g = \frac{2(s - ut)}{t^2}$; 32·2.

9. 2·7 in. to 2 sig figs. Increase is approx 2·4 h.p.
10. (a) $Q = \pm d\sqrt{2g(E - d)}$. (b) $R_0 = 82·3$.
(c) $r = \dfrac{2t^2}{2t - p}$. (d) $i = 5\frac{5}{16}$, $I = -1\frac{3}{16}$.
11. (a) $m = \dfrac{Mgt^2}{4\pi^2 a - 2gt^2}$. (b) (i) $p^{2b - a - 3c}$; (ii) $\dfrac{x^3}{y^3}$.
12. (a) $r = \dfrac{E}{i} - \dfrac{R}{n}$. (b) $n = \dfrac{iR}{E - ir}$.
13. (a) $s = ut + \frac{1}{2} ft^2$; dist. $= -57\frac{9}{40}$ ft.
(b) $n = \dfrac{360}{180 - \theta}$; $n = 24$.
14. 0·063° to 2 sig figs.

EXERCISE VII

p. 159.

Numerical answers inadequate.

EXERCISE VIII

Section A

p. 187.

1. (1) a^{10}. (2) b^{12}. (3) x^{12}. (4) $\frac{3}{32}x^{10}$. (5) $2^7 = 128$. (6) $3^7 = 2187$.
2. (1) a^4. (2) c^5. (3) x^{12}. (4) $2^6 = 64$.
3. (1) x^6. (2) a^3. (3) $\dfrac{1}{a}$. (4) x^7.
4. (1) a^{14}. (2) x^{12}. (3) $16b^{16}$. (4) $2^8 = 256$. (5) $10^6 = 1,000,000$. (6) $27a^6$. (7) $\frac{1}{32}x^{20}$. (8) $3^9 = 19,683$.

Section B

p. 188.

1. $\sqrt{3}$, $\frac{1}{4}$, $\dfrac{3}{a^2}$, 1, $\dfrac{1}{\sqrt{2}}$, 3, $3a^2$, $\sqrt{64}$, $\dfrac{1}{10^3} = 0·001$.
2. (1) 5·656. (2) 27. (3) $\frac{1}{10}$. (4) $a^{17/12}$. (5) 2·828. (6) 316·2.
3. (1) 4. (2) 125. (3) 1000. (4) $\dfrac{1}{5^6} = \frac{1}{15625}$. (5) 16. (6) 31·62.

4. (1) 4. (2) $\frac{27}{8}$. (3) 4. (4) $\frac{1}{6}$. (5) 8. (6) $\frac{1}{32}$.
5. 5·656. 6. (1) $\sqrt[6]{a^5}$. (2) 316·2.
7. (1) 5·656. (2) 22·62. (3) 15·59. (4) 1·190. (5) $\frac{1}{2}$. (6) 2·78. 8. 4·64. 9. 6·75.

Section C

p. 189.

1. 1, 3, 4, 2, 0, 5, 1, 3, 0, 2.
2. (1) 0·6990, 1·6990, 2·6990, 4·6990.
(2) 0·6721, 2·6721, 4·6721.
(3) 1·7226, 0·7226, 2·7226.
(4) 2·9767, 0·9767, 4·9767.
(5) 0·7588, 1·9842, 3·8433.
3. (1) 446·7, 44670, 44·67. (3) 4·714, 471·4, 471400.
(2) 87·70, 8770, 8·770. (4) 2628, 5·229, 114·0.

Section D

p. 189.

1. 344·6.	7. 1·589.	12. 2650.	17. 1·656.	22. 2·786.
2. 276·4.	8. 222·8.	13. 3·137.	18. 1436.	23. 5·002.
3. 1397.	9. 14·22.	14. 728·8.	19. 1·359.	24. 1·546.
4. 5977.	10. 13·56.	15. 2·172	20. 1·695.	25. 62·83.
5. 2·396.	11. 851·3.	16. 104·6.	21. 2·321.	26. 530·7.
6. 6·997.				

Section E

p. 190.

1. (1) 0·4469, $\bar{1}$·4469, $\bar{2}$·4469. (3) $\bar{3}$·9904, $\bar{4}$·9904, $\bar{1}$·9904.
(2) 0·6298, $\bar{1}$·6298, $\bar{3}$·6298. (4) $\bar{2}$·8097, $\bar{1}$·8097, $\bar{4}$·8097.
2. (1) $\bar{2}$·7771. (3) $\bar{3}$·9011. (5) $\bar{1}$·7538.
(2) $\bar{4}$·6749. (4) $\bar{5}$·9673. (6) $\bar{2}$·9023.
3. (1) 0·2159. (3) 0·03070. (5) 0·5940.
(2) 0·0007454. (4) 0·0004402. (6) $2·482 \times 10^{-8}$.
4. (1) $\bar{4}$·6037. (2) $\bar{2}$·7126.
5. (1) $\bar{2}$·5926. (3) $\bar{1}$·6597.
(2) 0·8263. (4) $\bar{2}$·4814.
6. (1) $\bar{1}$·7464. (3) $\bar{3}$·8910. (5) $\bar{1}$·1958.
(2) $\bar{4}$·8368. (4) $\bar{1}$·3673. (6) $\bar{4}$·7913.
7. (1) $\bar{2}$·6856. (3) $\bar{1}$·7754. (5) $\bar{2}$·0254.
(2) $\bar{1}$·07155. (4) $\bar{1}$·1463. (6) 0·5619.
8. (1) $\bar{1}$·7399. (3) $\bar{2}$·7726. (5) $\bar{1}$·7266.
(2) $\bar{1}$·7127. (4) $\bar{2}$·5598. (6) $\bar{3}$·8973.

SECTION F

p. 191.

1. 15·42.
2. 0·3285.
3. 0·01529.
4. 5·699.
5. 0·6116.
6. 0·03239.
7. 0·04903.
8. 0·1600.
9. 85·23.
10. 0·8414.
11. 0·1226.
12. 1·197.
13. 0·07115.
14. 1·826.
15. 1·457.
16. 3·558.
17. 5·471.
18. 0·1014.
19. 0·1429.
20. 9·399.
21. 483·2
22. 0·3817.
23. 34·45.
24. 10·87.
25. 6·944.

SECTION G. MISCELLANEOUS

p. 192.

1. (*a*) 9; 8; 64×10^5. (*b*) (i) 28·65; (ii) 0·6127.
2. (*a*) 1·035. (*b*) 1·603. (*c*) 1·238. (*d*) 1.
3. (i) 1·676. (ii) 375·9.
4. (*a*) (i) 195·7; (ii) 19·58. (*b*) $x = 5\frac{1}{2}$.
5. (*a*) 19·5. (*b*) 0·194. (*c*) 33·8. (*d*) 1·38.
6. $a^{(m+n)}$; $a^{(m-n)}$; a^{mn}.
7. (*a*) $10^{0·3010}$, $10^{1·6990}$, 10^2, $10^{5·0969}$. (*b*) 302, 2·512. (*c*) 2.
8. (*a*) 3. (*b*) 3.
9. 3·499, 1·6232, 0·4771, $10^{0·6990}$.
10. (1) $10^{0·2430}$, $10^{2·2430}$, $10^{1·656}$. (2) 69·31. (3) 50·06.
11. 102·2. 12. 7·745. 13. 1·441. 14. 13·5. 15. 6·4219.
16. 273 sq in. 17. 516·4. 18. 1·035.
19. (*a*) 1·663. (*b*) 1·794. (*c*) 1·421. (*d*) 109·5.
20. (*a*) 94·9. (*b*) 200.

EXERCISE IX

Angles

p. 216.

1. (1) 990°. (2) 1500°. (3) 12·32 P.M.
3. $22\frac{1}{2}°$, 56·25°, 101·25°.
4. $\angle$ABE = 45°, $\angle$BED = 135°, ABCD is a trapezium.

Theorem of Pythagoras

p. 216.

5. 23·43 miles. 7. 1·735 in. 8. 1·7 in.
10. 4·33 in. 12. 42·7 ft. 13. 311·1 yd.

Similar Figures

p. 217.

14. 22·25 in.
15. DE = 1·92 in., AE = 2·56 in.
16. FE = 89·44 yd, AE = 178·88 yd.
18. BD = 7·96 in., DC = 4·21 in., AD = 15·02 in.
19. 34·61 sq in. 20. $\frac{1}{4}$ in. to 1 mile, $3\frac{15}{16}$ in.
21. 3·69 in., 3·185 sq in.

Miscellaneous

p. 218.

22. 2·887 in. 23. 30°, 60° and 90°.
25. 72°; 144°. 26. $\alpha = 135°$. 27. (i) $25\frac{1}{7}$ in. (ii) $4\frac{4}{21}$ in. (iii) 90°.
29. Approx 24·5 ft and 17·9 ft.
30. 60 ft. 31. 9·32 in., 2·68 in. 32. 1·6 in.
33. 17·6 sq in., $1\frac{1}{2}$ in. represents 5 ft.
34. 3·664 in. and 0·836 in.; 1·75 in. and 1·75 in.
35. $p^2 = 4(x + y)^2$, $4d^2 = 4(x^2 + y^2)$, 120 sq in.

EXERCISE X

SECTION A

p. 246.

1. $\tan ABC = \frac{AC}{CB} = \frac{CD}{DB} = \frac{CQ}{QD} = \frac{DQ}{QB} = \frac{AD}{CD}$.

   $\tan CAB = \frac{CB}{AC} = \frac{DB}{CD} = \frac{QD}{CQ} = \frac{QB}{DQ} = \frac{CD}{AD}$.
2. 31° nearly.
3. tan ABC = $\frac{4}{3}$, tan CAB = $\frac{3}{4}$.
4. (1) 0·3249. (2) 0·9325. (3) 1·4826. (4) 3·2709. (5) 0·2549. (6) 0·6950.
5. (1) 0·1635. (2) 0·6188. (3) 0·8122. (4) 1·3009. (5) 2·1123.
6. (1) 28° 36′. (2) 61° 18′. (3) 70° 30′. (4) 52° 26′. (5) 33° 51′. (6) 14° 16′.
7. 29·8. 8. 46° 18′. 9. 67° 23′, 67° 23′, 45° 14′. 10. 52·1 ft.
11. 211 ft. 12. 213 ft approx. 13. 69·3 in. approx.
14. 148·3 ft; 25°. 15. 37°; 53° approx.

Section B

p. 247.

1. $\sin ABC = \dfrac{AB}{AC} = \dfrac{DQ}{DB} = \dfrac{CB}{CB} = \dfrac{CQ}{CD} = \dfrac{AD}{AC}.$

$\sin CAB = \dfrac{CB}{AB} = \dfrac{QB}{DB} = \dfrac{DB}{CB} = \dfrac{DQ}{CD} = \dfrac{CD}{AC}.$

$\cos ABC = \dfrac{CB}{AB} = \dfrac{QB}{DB} = \dfrac{DB}{CB} = \dfrac{DQ}{CD} = \dfrac{CD}{AC}.$

$\cos CAB = \dfrac{AC}{AB} = \dfrac{DQ}{DB} = \dfrac{CD}{CB} = \dfrac{CQ}{CD} = \dfrac{AD}{AC}.$

2. Cosine is 0·1109, sine is 0·9939.
3. Length is 5·14 in. approx., distance from centre 3·06 in. approx.
4. Sines 0·6 and 0·8, cosines 0·8 and 0·6.
5. (1) 0·2521. (2) 0·7400. (3) 0·9353.
6. (1) 29° 48′. (2) 30° 46′. (3) 52° 14′.
7. (1) 0·9350. (3) 0·4594. (5) 0·1863.
(2) 0·7149. (4) 0·7789. (6) 0·5390.
8. (1) 57° 47′. (3) 69° 14′. (5) 37° 43′.
(2) 20° 39′. (4) 77° 27′. (6) 59° 4′.
9. 10° 5′, 34·2 ft. 11. 7·34 in.; 37° 48′; 52° 12′.
10. 18° 26′. 12. 0·6, 0·8.

Section C

p. 248.

1. (1) 1·7263. (3) 1·3589. (5) 1·2045.
(2) 1·1576. (4) 1·6649. (6) 0·3528.
2. (1) 60° 37′. (2) 64° 45′. (3) 69° 18′.
4. 4·82 in. 5. 22° 37′, 67° 23′. 6. 2·87 in.

Section D

p. 249.

1. 35° 1′, 54° 59′, 28·6. 2. $a = 45{\cdot}43$, $c = 58{\cdot}36$.
3. A = 44° 8′, b = 390 ft approx. 4. 69° 31′, 60°, 1·42 ft.
5. 9·7 ft, 50° 54′. 6. 58·6 sq in.
7. (*a*) 0·68 cm. (*b*) 70°. 8. 3·64 m., 45° W. of N.

Section E. Miscellaneous

p. 250.

1. 1·7100 in.; 2·5000 in.; 3·5355 in.
2. 0·9811; (i) $\frac{20}{29}$, $\frac{21}{20}$, $\frac{21}{29}$; (ii) 0·4602, 1·4176; (iii) 31° and 211°. 3. 570 ft; 62° 54′.
4. 70°. 5. 23°. 6. 0·2 in. dia.
7. 20 ft 3 in. approx. 42 ft per sec to 2 sig figs.
8. (*a*) Sine is $\frac{12}{13}$; cosine is $\frac{5}{13}$. (*b*) 1450 ft.
9. 2·06 in. 10. 0·862 in. 11. 1·73 in. to 3 sig figs.
12. (*a*) 29°; 9·67 ft. (*b*) (i) 87·1; (ii) 1·1022.
14. 38·7 ft. 15. 9° 12′.
16. (*a*) 0·075. (*c*) −0·5 approx. (*d*) 60°, 66° 24′, 300°, 293° 36′.
17. Approx 154,000 gal. 18. 52° 3′, 39° 44′.
19. AC = 220 yd; 190 yd. 20. 12°.
21. (1) 893·2 sq ft. (2) 27·8 sq in.
22. 7·3. 23. −0·217.

EXERCISE X1

Section A

p. 267.

1. (*a*) 35·19 in. (*b*) 109·3 ft. (*c*) 18·22 cm.
2. (*a*) 117·9 ft. (*b*) 9·10 in. (*c*) 4·84 cm.
3. 9 ft 11·4 in. 4. 24,880 miles. 5. 3·05 in., 8·46 in.
6. (1) 30·55 in. (2) 28·68 in.
7. 2·036 in., 5·504 in. 8. 49·5 sq in. 9. 336.
10. 13·1 in. 11. (1) 6π in. (2) $\pi(12D - d)$ in.

Section B

p. 268.

1. (*a*) 6·16 sq cm. (*b*) 45·4 sq in. (*c*) 490·9 sq ft.
2. (*a*) 2 in. (*b*) 19·95 ft. (*c*) 3·57 cm. (*d*) 3·23 in.
3. 28·52 in. 4. 200·5 cm.
5. (*a*) 51·33 sq in. (*b*) 372·3 sq cm. (*c*) 1223 sq ft.
6. $127\frac{3}{7}$ sq ft. 7. 11·17 sq in., 67·37 sq in.
8. 5·515 lb. 9. £1 2*s*. 1*d*.

Section C

p. 268.

1. (*a*) 4·75. (*b*) 2·545. 2. (*a*) 286° 29′. (*b*) 13° 24′. (*c*) 89° 23′.
3. (1) 9·1 ft per sec. (2) 19·62 ft per sec. (3) 12·14 cm per sec.

4. 1·2 ft.
5. (1) 41·89 in. (2) 15 in.
6. 2·094, 5·236 ft per sec.
7. 2·2 radians, 126° 3′.
8. $\frac{11}{18}$ radians, 35°, 25 ft 8 in.
9. 0·2194.

Miscellaneous

p. 270.

1. $b = 0{\cdot}337$ in.
2. (*a*) 110° 20′, 249° 40′. (*b*) 108° 12′, 288° 12′.
   (*c*) 35° 46′, 91° 14′.
3. 210°; 330°.
4. 32·1 in.
5. (i) 10 ft. (ii) 18·5 ft to 3 sig figs.
   (iii) 44·7 sq ft to 3 sig figs.
6. $12\frac{1}{4}$ sq in.
7. (*a*) 45°, 540°. (*b*) $12\frac{1}{2}$ min past 2.
8. 32·73 min to 4 sig figs.
9. (*a*) (i) 31·42 ft per sec to 4 sig figs; (ii) $9\frac{3}{7}$ sq ft.
   (*b*) 7·13 ft per sec to 3 sig figs.
10. (i) 30°, 45°, 360°, 76·8°. (ii) 0·6415 radians. (iii) 2·3 sq in.
11. (*b*) $d = h + \frac{W^2}{4h}$. (*c*) 2 in.
12. 7 ft; 60 ft 6 in.

EXERCISE XII

Section A

p. 294.

1. 189 cu in., 63·8 lb.
2. 16·39 cc.
3. 400 cu ft.
4. $1\frac{1}{2}$ cu ft.
5. 149·6 gal.
6. 600 lb.
7. 16 sq ft.
8. 9·5 tons.
9. 6 ft 8 in.
10. $2691\frac{1}{2}$ cu in.
11. 6·5 cu in.
12. 180 gal.

Section B

p. 295.

1. (*a*) 224·7 sq in. (*b*) 15·08 sq ft.
2. £4 14*s*. 3*d*.
3. 1925 sq ft.
4. $1026\frac{2}{3}$ sq ft.
5. 76·98 sq ft.
6. (*a*) 42·41 sq in. (*b*) 6008 cc.
7. (*a*) 17·72 cu ft. (*b*) 2870 cu in.
8. 459·5 lb.
9. 7·78 cm.
10. 2227 metres.
11. 4054 lb.
12. 8·17 cu in.

Section C

p. 297.

1. (*a*) 144 cu in. (*b*) 184·4 sq in.
2. 15·6 cc.
3. (*a*) 190·9 cu in. (*b*) 40·72 cu ft.
4. (*a*) 205·9 sq in. (*b*) 78·78 sq ft.
5. 233·4 cu ft.
6. (*a*) 4·98 cu in. (*b*) 12·32 sq in.
7. 0·6234 sq in.
8. 40·14 sq in.
9. $50\frac{5}{8}$ cu in.
10. 42·19 lb.
11. (*a*) 72·39 sq cm. (*b*) 394·1 sq in.
12. (*a*) 17·16 cc. (*b*) 310·3 cu in. (*c*) 65·45 cu ft.
13. £3 9*s*. 1*d*.

Section D. Miscellaneous

p. 298.

1. Method is important.
2. 15 6 lb to 3 sig figs.
3. 2·914 ft.
4. (*a*) 204 sq ft. (*b*) 3050 cu ft, and 2040 sq ft, all to 3 sig figs.
5. 2650 lb to 3 sig figs.
6. (i) $2\frac{3}{8}$ in. approx. (ii) $12\frac{1}{2}$ m.p.h.
7. 78 metric tons.
8. 1·8 gal to 3 sig figs.
9. $48Lt(x + t) + 2tx^2$ cu in.; $48Ltz(x + t) + 2tx^2z$ lb.
10. $36\frac{1}{3}$ in.
11. 98·4 cu in.; 29 lb, both to 3 sig figs.
12. To 3 sig figs —155 lb plus allowance for runners, etc. and waste.
13. (*a*) 7 ft 6 in. (*b*) 12 ft 3 in. (*c*) 73° 18′.
14. (i) 12 in. (ii) 981 cu in. (iii) 377 sq in. All to 3 sig figs.
15. 8·47 cm; 7·47 cm to 3 sig figs.
16. 845 sq. in. to 3 sig figs.
17. 2400 cu ft per hr to 3 sig figs.
18. (*a*) 2 ft $1\frac{1}{3}$ in. (*b*) 129 gal per min.
19. (*b*) 1570 sq ft.
20. (i) 3 : 2. (ii) 9 : 4.
21. 15·71 cu ft.
22. 2·76 lb.
23. 117·8 gal.
24. 5·5 lb.
25. (*a*) $V = t(L^2 - \pi r^2)$.
    (*b*) 60·82%.
26. 7·64 lb, 2·34 lb.
27. 188 lb.
28. 398·2 cu in., 119·5 cu in., 4·33.
29. 0·303 lb, 30·3 lb.
30. 146·32 lb, 2,593 lb.
31. 1 : 1.
32. 952·8 lb.
33. $22h^2$, 3 in.; 135 cu in.
34. 402·9 lb.
35. 86·94 gal.
36. 65·5 cm.

37. $\frac{\pi^2 d^3}{2}$, $\frac{l^3}{16\pi}$, 6 in. 38. 1·21 in.
39. 0·05 sq cm, 0·252 cm. 40. 115·8 lb.
41. 21·83. 42. 10 lb.

EXERCISE XIII

p. 318.

1. 1·94 ohms. 2. 160 c.p.
3. (*a*) 27·6 h.p. (*b*) 93*s*. 9*d*.
4. (i) $3\frac{1}{16}$ in. (ii) 0·32 in. 5. 0·5 in.
6. (*b*) 11 in. (*c*) 11,520 sq yd.
7. 20 in. and 32 in.; 9 : 25 : 64.
8. $d = 2\sqrt[3]{\frac{H}{12}}$, 40·5. 9. 1·038 secs.
10. 4·08 ft. 11. 2·8 in.
12. 65 lb per sq in., 2·16 cu ft. 13. 324·0 ohms.
14. 237·2 ft. 15. 2·048 ohms.
16. 8 tons. 17. $31\frac{1}{4}$ cwts.
18. 19*s*. 19. 15·88 knots.

EXERCISE XIV

p. 340.

1. (1) 3·9 in. (2) 1·77 sq in. 2. $-4, \frac{1}{2}$.
3. $3, -2$. 4. $-3, 1, -4$. 5. 0·283, 0·387.
6. $-$ 0·66, 2·66.
7–11. Answers embodied in appropriate graphs.
12. (1) $12\frac{1}{4}$. (2) $6, -1$. 13. 3, 5·45, 0·55.
14. $3, -1\frac{1}{2}$. 15. 2·08, $-$ 2·88.
16. (1) $17{\cdot}5 - 5x$. (2) $17{\cdot}5x - 5x^2$. (3) $1\frac{3}{4}$ in.
17. $11\frac{3}{5}$ in., $16\frac{2}{5}$ in. 18. $\frac{1}{2}$. 19. $\pm$ 2·83.
20. 1·36. 21. 18·8, 3·13. 22. 15·2 tons. 23. 30·7 ft lb.

EXERCISE XV

Section A

p. 353.

1. $x + 2$. 2. $x - 4$. 3. $x - \frac{1}{2}$
4. $x - \frac{1}{3}$. 5. $x + 20$. 6. $x - 0{\cdot}1$.
7. $R - 2{\cdot}5$ 8. $\frac{1}{x} - \frac{1}{2}$. 9. $\frac{1}{a} - \frac{2}{3}$.
10. $2x - 3a$. 11. $5 - x$. 12. $\frac{1}{a} + \frac{1}{b}$.

Section B

p. 353.

1. 14, 4 2. $7, -15$. 3. $-$ 0·35, $-$ 5·65.
4. 5, 0. 5. $4\frac{1}{3}, -\frac{5}{3}$. 6. $-$ 5·16, $-$ 1·84.
7. $7, -3$. 8. $2, -12$

Section C

p. 354.

1. $6\frac{1}{4}$. 2. $\frac{1}{4}$. 3. $20\frac{1}{4}$.
4. $4a^2$. 5. $30\frac{1}{4}$. 6. $1\frac{9}{16}$.
7. $\frac{1}{36}$. 8. $\frac{4}{b^2}$. 9. 0·04.
10. 0·5625.

Section D

p. 354.

1. $3, -4$. 2. $11, -2$.
3. $-9, 2$. 4. 7, 5.
5. $7, -2$. 6. 2·2, $-$ 3·2.
7. $11, -2$. 8. 2·5, $-$ 9·5.
9. 0·49, $-$ 0·82 10. 0·5, $-$ 0·9.
11. 1·48, $-$ 1·08. 12. 1·655, 0·145.
13. 0·85, $-$ 0·75. 14. $2\frac{1}{2}, -1$.
15. $-\frac{2}{3}, -5$. 16. $3, -\frac{1}{2}$.
17 $-5, -\frac{1}{2}$ 18. 3·25, $-$ 0·92.
19. $-2, -\frac{1}{2}$. 20. $1, -5\frac{1}{2}$.
21. 1·43, 0·23. 22. 2·85, $-$ 0·35.
23. 1·744, $-$ 0·344. 24. $5, -1$.
25. 2·732, $-$ 0·732 26. 3·19, 0·314.
27. 13·14, $-$ 1·14 28. 6·59, $-$ 7·59.
29. 6·53, $-$ 1·53.

Section E

p. 355.

1. $12, -3$. 2. $-4, -3$. 3. $2\frac{1}{2}, -1$.
4. $11, -9$. 5. $1\frac{2}{3}, -3\frac{1}{2}$. 6. $\frac{1}{2}, -\frac{1}{4}$.
7. 0·1, $-$ 0·2. 8. 0·7, $-$ 1·2. 9. $\frac{2}{3}, -1\frac{1}{3}$.
10. $1, -2$.

Section F

p. 355.

1. $x^2 - x - 6 = 0$.
2. $x^2 - 9x + 20 = 0$.
3. $x^2 - \frac{1}{4} = 0$.
4. $x^2 - 3{\cdot}9x + 3{\cdot}5 = 0$.
5. $x^2 - 0{\cdot}6x - 0{\cdot}72 = 0$.
6. $x^2 - ax - 2a^2 = 0$.

Section G

p. 355.

1. (i) $x = \frac{2}{3}$ or $\frac{3}{2}$. (ii) $A = \frac{20}{7}$, $B = 175$, $R = 23\frac{3}{7}$.
2. $2\frac{1}{3}$ or -6.
3. 540 ml.
4. (*a*) (i) $x = 6$; (ii) $x = 3$, $y = -2$. (*b*) 3 m.p.h.
5. (*a*) $x = 3\frac{1}{2}$ or $-4\frac{1}{3}$. (*b*) 24 m.p.h.
6. (*a*) (i) 2·781 or 0·719; (ii) 0·646 or − 4·646.
   (*b*) 8·944 × 17·888.
7. (*b*) 8·96 or 0·37.
8. 0 or 3 in.
9. 4 ft or 6 ft.
10. 1·385, 0·985.
11. 3 yd.
12. 2 or $-1\frac{1}{2}$.
13. 9·32 in., 2·68 in.
14. 2 in.
15. 4·5 ft per sec.
16. 21·37, 27·37.
17. Depth 7 in. very nearly.
18. $r = 3$ in.
19. 6 in., 2 in.
20. 88·23.
21. 12 in., 9 in.

EXERCISE XVI

p. 367.

1. $6{\cdot}4_{86{\cdot}5^\circ}$.
2. $8{\cdot}8_{50^\circ}$.
3. 11_{77°.
4. $5{\cdot}6_{351{\cdot}5^\circ}$, $2{\cdot}6_{19^\circ}$.
5. 88·4° N. of E., 16·5 m.p.h.
6. 6·7 lb, 56·5° with OX.
7. 26 ml South; 15 ml West
8. (i) (*a*) $34{\cdot}9_{51^\circ}$. (*b*) horizontal is 22, vertical is 27·2.
   (ii) 21·2 knots, N. 53° 37′ E.
9. (i) $5{\cdot}37_{91^\circ 36'}$. (ii) 14·93 lb at 11° 50′ on side of 10 lb pull (pole held vertical).
10. 5 lb, 60° S. of W.
11. $1{\cdot}73_{30^\circ}$.
13. $7{\cdot}9_{35{\cdot}6^\circ}$.
14. 4·36 ft per sec, 36·6° with horizontal.
15. 6·94 in., 38·5° N. of E.

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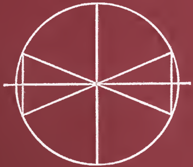

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W. E. FISHER. O.B.E. D.Sc.